

Notation for NE 112 *Linear algebra for nanotechnology engineering*

By Douglas Wilhelm Harder.

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This list of notations is not comprehensive. If you believe there is a notation missing, or if you come across a notation that is not used our course but think that term should appear in this list for information purposes, please contact the instructor.

^{def} $a = expression$	The left-hand side is being defined as the expression on the right-hand side.
$f : x \mapsto expression$ or $f : (x, y, z) \mapsto expression$	The left-hand side is defined as a function that takes a variable x or three variables x , y , and z and maps the to the expression on the right-hand side. For example, if $p : x \mapsto x^2 - 1$ then $p(3) = 8$ and $p(a + b) = a^2 + 2ab + b^2 - 1$.
$A, B, C, \dots, A_0, A_1, \dots$	Generally, a linear operator; usually a matrix.
$A : \mathcal{U} \rightarrow \mathcal{U}$	A is a linear operator that maps vectors in the vector space $\mathcal{U}$ to vectors in that same vector space.
$A : \mathcal{U} \rightarrow \mathcal{V}$	A is a linear map that maps vectors in the vector space $\mathcal{U}$ to vectors in the vector space $\mathcal{V}$.
$\mathbf{C}$	The field of complex numbers.
$\mathbf{C}^n$	The vector space of n -dimensional complex vectors.
$\mathbf{C}[x]$	The vector space of polynomials in the variable x where the coefficients are complex numbers.
$\mathbf{F}$	Either the field of real numbers $\mathbf{R}$ or the field of complex numbers $\mathbf{C}$. This is used when a property, example or result does not depend on the scalars being real or complex numbers.
$\mathbf{F}^n$	Either the vector space $\mathbf{R}^n$ or the vector space $\mathbf{C}^n$. This is used when a property, example or result does not depend on the entries being real or complex numbers.
$\mathbf{F}[x]$	Either the vector space $\mathbf{R}[x]$ or the vector space $\mathbf{C}[x]$. This is used when a property, example or result does not depend on the polynomials having either real or complex coefficients.
I	The identity operator $I : \mathcal{U} \rightarrow \mathcal{U}$ where $I\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{U}$.
I_n	The $n \times n$ identity matrix, a matrix with each diagonal entry being 1 and all off-diagonal entries being zero.
$\mathbf{R}$	The field of real numbers.
$\mathbf{R}^n$	The vector space of n -dimensional real vectors.
$\mathbf{R}[x]$	The vector space of polynomials in the variable x where the coefficients are real numbers.
$R_{Ri \leftrightarrow Rj}$	The row operation swapping Rows i and j . This can either simply specify the row operation that is being performed, or it may represent an $n \times n$ matrix which is the identity matrix I_n with Rows i and j swapped.
$R_{Ri \leftrightarrow Rj}$	
$R_{Ri \leftrightarrow Rj}$	
$\mathbf{u}, \mathbf{v}, \mathbf{w}, \dots, \mathbf{u}_0, \mathbf{u}_1, \dots$	Generally, a finite-dimensional vector
x, y, z	When discussing 3-dimensional vectors, the three coordinates.
x, y	Generally real values, often used with $y = f(x)$ where f is a real-valued function of a real variable.
x, y	When discussing infinite sequences or infinite discrete signals, these vectors are represented by lower-case italicized letters.
w, z	Generally complex values.
f, g	Generally functions of either a real or complex variable mapping those values to the same field.

p, q	Generally, polynomials with coefficients that are either real or complex.
$\alpha, \beta, \gamma, \delta$	When describing a general field $\mathbf{F}$ (either real or complex), scalars are often represented by Greek letters. When discussing complex numbers, the real and imaginary components are usually represented as Greek letters, so $z = \alpha + \beta j$.
$A: \mathcal{U} \rightarrow \mathcal{U}$	A is a linear operator that maps vectors in the vector space $\mathcal{U}$ to vectors in the vector space $\mathcal{V}$.
$A: \mathcal{U} \rightarrow \mathcal{V}$	The property that you can perform a sequence of the same operation in any order; for example, in calculating $x + y + z$, it does not matter if you calculate $(x + y)$ first and then add z , or calculate $(y + z)$ first and then add this to x . Addition and multiplication of scalars in a field are both associative; vector addition is associative; linear map addition (matrix addition) and linear map composition (matrix-matrix multiplication) are both associative; however, the cross product is not: $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ will usually give a different result if you calculate $\mathbf{u} \times \mathbf{v}$ first as opposed to calculating $\mathbf{v} \times \mathbf{w}$ first. Other operations that are not associative are subtraction and scalar division: $(\alpha - \beta) - \gamma \neq \alpha - (\beta - \gamma)$, $(\mathbf{u} - \mathbf{v}) - \mathbf{w} \neq \mathbf{u} - (\mathbf{v} - \mathbf{w})$ and $(\alpha \div \beta) \div \gamma \neq \alpha \div (\beta \div \gamma)$.
$\langle \mathbf{u}, \mathbf{v} \rangle$	The inner product of two vectors, $\mathbf{u}$ and $\mathbf{v}$ that come from the same vector space.
$\ \mathbf{u}\ $	The norm of the vector $\mathbf{u}$, where the specific norm is not significant to the property, example or result being discussed.
$\ \mathbf{u} - \mathbf{v}\ $	Technically the norm of the vector $\mathbf{u} - \mathbf{v}$, but more usually interpreted as the distance between $\mathbf{u}$ and $\mathbf{v}$ using whichever norm is specified.
$\ \mathbf{u}\ _1$	The 1-norm. For finite-dimensional vectors, the sum of the absolute values of the entries of the vector.
$\ \mathbf{u}\ _2$	The 2-norm, or the norm induced by the inner product. For finite-dimensional vectors, the square root of the sum of the absolute values squared of the entries of the vector.
$\ \mathbf{u}\ _\infty$	The infinity-norm. For finite-dimensional vectors, the maximum entry in absolute value.
$\mathbf{u} \times \mathbf{v}$	Generally the cross product of the two 3-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$. Occasionally, it will be used to denote multiplication when juxtaposition (xy) or a small dot ($0 \cdot y$) may be ambiguous or confusing.
BA	The composition of the linear map B and A where $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{W}$. The resulting linear map $BA: \mathcal{U} \rightarrow \mathcal{W}$. If all three vector spaces are finite-dimensional vectors spaces (real or complex), this represents matrix-matrix multiplication. Important: a non-square matrix is generally described as being $m \times n$, and this maps the vector space $\mathbf{F}^n$ to $\mathbf{F}^m$. Many text books use $n \times m$ matrices; however, this is less desirable, as when we get to matrix-matrix products, we have $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ and $B: \mathbf{F}^m \rightarrow \mathbf{F}^\ell$, so $BA: \mathbf{F}^n \rightarrow \mathbf{F}^\ell$.
$A\mathbf{u}$	The product of the matrix A and the vector $\mathbf{u}$ where $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, $\mathbf{u} \in \mathbf{F}^n$ and $A\mathbf{u} \in \mathbf{F}^m$. If the vector is a linear combination of vectors, parentheses will be used, so for example, $A(\alpha \mathbf{u} + \beta \mathbf{v})$.
u_k	The entries of a finite-dimensional vector $\mathbf{u} \in \mathbf{F}^n$. The entries will always have the same letter, only the entries will be italicized while the vector is bold.
$a_{i,j}$	The entries of a matrix $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$. The entries will always have the same letter, only the entries will be lower-case italicized while the matrix is upper-case bold.
z^*	The complex conjugate of a complex number z .
A^*	The adjoint of a linear map A . If A is a real matrix, the adjoint is the transpose. If A is a complex matrix, the adjoint is the conjugate transpose.
A^T	The transpose of a real matrix A . We will never take <i>just</i> the transpose of a complex matrix; we will only take the conjugate transpose of a complex matrix, and that will always be denoted as A^* .
$\det(A)$	The determinant of a real or complex matrix A .
$\text{tr}(A)$	The trace of a real or complex matrix A .

λ	Generally, an eigenvalue of a linear operator.
θ and ϕ	Generally, an angle.
j, i	The imaginary unit or the square root of -1 . Engineers use j while mathematicians use i . We will never use i in this course, but programming languages like Matlab, while accepting j will usually display the output with i .
$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$	The three canonical basis vectors in $\mathbf{R}^3$.
$\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_n$	The n canonical basis vectors in $\mathbf{F}^n$.
$\Im m(z)$	The imaginary component of the complex number z .
$\sum_{j=m}^n a_k$	The sum of the values a_m through a_n , inclusive.
$[a, b]$	The closed interval on the real line including all real numbers from a to b including both a and b .
(a, b)	The open interval on the real line including all real numbers from a to b , but not including either a or b .
$(a, b]$	The half-open (or half-closed) interval on the real line including all real numbers from a to b including b but not including a .
$[a, b)$	The half-open (or half-closed) interval on the real line including all real numbers from a to b including a but not including b .
x^{-1}	The reciprocal or multiplicative inverse of the scalar x . It is assumed $x \neq 0$.
A^{-1}	The inverse of the linear operator A . It is assumed that A is invertible.
$A = A^*$ or $A = A^T$	The equation specifying that the linear operator A is self-adjoint, meaning that if it is a real square matrix, then A is symmetric and equals its transpose, and if it is a complex matrix, then A is conjugate symmetric and equals its conjugate transpose.
$A = -A^*$ or $A = -A^T$ or $-A = A^*$ or $-A = A^T$	The equation specifying that the linear operator A is skew-adjoint, meaning that if it is a real square matrix, then A is skew-symmetric, and if it is a complex matrix, then A is conjugate skew-symmetric.
$z = z^*$	The equation specifying that the complex number z equals its complex conjugate, which means that z is a real number.
$z = -z^*$ or $-z = z^*$	The equation specifying that the complex number z equals its complex conjugate negated, which means that z is an imaginary number.
$A^{-1} = A^*$ or $A^{-1} = A^T$	The equation specifying that the linear operator A is isometric, meaning that $\ A\mathbf{u}\ _2 = \ \mathbf{u}\ _2$ for all vectors $\mathbf{u}$.
$\mathbf{0}$ or $\mathbf{0}_{\mathcal{U}}$	The zero vector in a given vector space $\mathcal{U}$.
$\mathbf{0}_n$	The zero vector in the finite-dimensional vector space $\mathbf{R}^n$ or $\mathbf{C}^n$.
$r\angle\theta$ or $r\angle\theta$ or $re^{j\theta}$	The polar representation of a complex number where $r \geq 0$ and generally $0 \leq \theta < 2\pi$ or $-\pi < \theta \leq \pi$.
$\text{proj}_{\mathbf{v}}(\mathbf{u})$	The projection of $\mathbf{u}$ onto the vector $\mathbf{v}$.
$\text{perp}_{\mathbf{v}}(\mathbf{u})$	The perpendicular component of $\mathbf{u}$ with respect to the vector $\mathbf{v}$, and equal to $\mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u})$.
$\mathbf{u} \perp \mathbf{v}$	$\mathbf{u}$ is orthogonal or perpendicular to the vector $\mathbf{v}$; that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
$\mathbf{u} \parallel \mathbf{v}$	$\mathbf{u}$ is parallel to the vector $\mathbf{v}$; $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ or $\langle \mathbf{u}, \mathbf{v} \rangle = \pm \ \mathbf{u}\ _2 \ \mathbf{v}\ _2$.
$A \sim B$	The matrix A is row equivalent or similar to a matrix B , meaning there exists a sequence of row operations that converts A to B and vice versa.
$\text{rank}(A)$	The rank of the matrix A , meaning the number of non-zero rows in any matrix that A is row-equivalent to that is also in row-echelon form.
$\text{range}(A)$	The range of the linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ where $\text{range}(A)$ must be a subspace of $\mathcal{V}$.
$\text{null}(A)$	The null space of the linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ where $\text{null}(A)$ must be a subspace of $\mathcal{U}$.
$\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$	All linear combinations of the vectors $\mathbf{u}_1$ through $\mathbf{u}_k$. It is assumed that all vectors come from the same vector space. The span is always a subspace of that vector space.

$\max\{x_1, \dots, x_k\},$ $\min\{x_1, \dots, x_k\}$	The maximum or minimum of the k real numbers x_1 through x_k .
$f(x) = f(-x)$	The equation specifying that the function f is even.
$f(x) = -f(-x)$ or $-f(x) = f(-x)$	The equation specifying that the function f is odd.
O	The zero map $O: \mathcal{U} \rightarrow \mathcal{V}$; the zero vector for linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$.
$O_{m,n}$	The $m \times n$ matrix or all zeros; the zero vector for matrices $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$.
$\mathcal{L}(\mathcal{U}, \mathcal{V})$	The vector space of all linear maps $A: \mathcal{U} \rightarrow \mathcal{V}$. To say that $A \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ is equivalent to saying $A: \mathcal{U} \rightarrow \mathcal{V}$.
$\mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$	The vector space of all $m \times n$ matrices with entries in the field $\mathbf{F}$. To say that $A \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ is equivalent to saying $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, which is equivalent to saying that A is an $m \times n$ matrix.
$\mathcal{L}(\mathcal{U})$	The vector space of all linear operators $A: \mathcal{U} \rightarrow \mathcal{U}$. In addition to being a vector space, this also has operator composition, and is therefore an algebra. Operator composition does not satisfy the conditions of scalar multiplication in a field. To say that $A \in \mathcal{L}(\mathcal{U})$ is equivalent to saying $A: \mathcal{U} \rightarrow \mathcal{U}$.
$\mathcal{L}(\mathbf{F}^n)$	The vector space of all $n \times n$ matrices with entries in the field $\mathbf{F}$. In addition to being a vector space, this also has matrix multiplication, and is therefore an algebra. Matrix multiplication does not satisfy the conditions of scalar multiplication in a field. To say that $A \in \mathcal{L}(\mathbf{F}^n)$ is equivalent to saying $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$, which is equivalent to saying that A is a square $n \times n$ matrix.