

A glossary for NE 112 *Linear algebra for nanotechnology engineering*

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This glossary is not comprehensive. If you believe there is a term missing, or if you come across a term that is not used our course but think that term should appear in this list for information purposes, please contact the instructor.

Any term in red and italics is not used in this course, but included for completeness, as you may come across such terms in any additional reading.

Absolute value	Given a real number x or a complex number z , the absolute value is a positive real number. For real $x > 0$, $ x = x$ and for $x < 0$, $ x = -x$. For complex $z = \alpha + \beta j$, the absolute value is $ z = \sqrt{\alpha^2 + \beta^2}$. The absolute value is zero if and only if the real or complex number is 0, otherwise, the absolute value is a real number greater than zero.
Additive identity	In a field, a 0 such that $\alpha + 0 = \alpha$ for all scalars α in the field, and every scalar has an additive inverse such that $\alpha + (-\alpha) = 0$.
Additive inverse	In a field, each scalar α has an additive inverse $-\alpha$ such that $\alpha + (-\alpha) = 0$. For rational and real numbers, the additive inverse of α is $-\alpha$, while for a complex $z = \alpha + \beta j$ is $-z = -\alpha - \beta j$. In a vector space, each vector $\mathbf{u}$ has an additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ where $\mathbf{0}$ is the zero vector in that vector space. The additive inverse of $\mathbf{u}$ is always found by multiplying the vector $\mathbf{u}$ by the scalar -1 .
Adjoint	Given a linear map A , the adjoint is that map A^* such that $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^*\mathbf{v} \rangle$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. For real matrices, the adjoint is the transpose of the matrix, and for complex matrices, the adjoint is the conjugate transpose of the matrix.
<i>Adjoint matrix</i>	See co-factor expansion. In linear algebra, there are two concepts of an adjoint. The one we use in this class, and the one used in quantum mechanics, is the adjoint of a linear operator. Given a matrix, the adjoint matrix is now usually referred to as the <i>classical</i> adjoint matrix, and is not appropriate for engineering, as the calculation thereof is exceptionally expensive and inefficient and subject to numerical error. If you ever come across the adjoint defined in terms of co-factor matrices, ignore this. Or, please consider transferring to any other university for the balance of your academic career.
Angle of a complex number	The angle of $z = \alpha + \beta j$ is defined as $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$ taking into account the signs of α and β . Also known as the phase of z . The angle of 0 is defined to be 0.
Angle between vectors	Given two vectors in $\mathbf{R}^2$, the angle between those vectors is the smallest angle between those vectors in the plane. In $\mathbf{R}^3$, the angle between the two independent vectors is the smallest angle between those vectors in the plane containing those two vectors. It happens that, in both these cases, the angle equals $\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\ \mathbf{u}\ _2 \ \mathbf{v}\ _2}\right)$, and thus, this more general definition is used to describe the angle between any two vectors in a given vector space. If $\mathbf{u} = \alpha \mathbf{u}$, if $\alpha > 0$, then the angle is 0° or 0 radians, and if $\alpha < 0$, then the angle is 180° or π radians. Two vectors are orthogonal if the angle is 90° or $\frac{1}{2}\pi$ radians.
Anti-symmetric matrix	Another term for a skew-symmetric matrix. This term is not used in this course.
Associativity	The property that you can perform a sequence of the same operation in any order; for example, in calculating $x + y + z$, it does not matter if you calculate $(x + y)$ first and then add z , or calculate $(y + z)$ first and then add this to x .

	Addition and multiplication of scalars in a field are both associative; vector addition is associative; linear map addition (matrix addition) and linear map composition (matrix-matrix multiplication) are both associative; however, the cross product is not: $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ will usually give a different result if you calculate $\mathbf{u} \times \mathbf{v}$ first as opposed to calculating $\mathbf{v} \times \mathbf{w}$ first. Other operations that are not associative are subtraction and scalar division: $(\alpha - \beta) - \gamma \neq \alpha - (\beta - \gamma)$, $(\mathbf{u} - \mathbf{v}) - \mathbf{w} \neq \mathbf{u} - (\mathbf{v} - \mathbf{w})$ and $(\alpha \div \beta) \div \gamma \neq \alpha \div (\beta \div \gamma)$.
Augmented matrix	When solving a system of linear equations $A\mathbf{u} = \mathbf{v}$ for the unknown vector $\mathbf{u}$ where A and $\mathbf{v}$ are given, a useful data structure is the augmented matrix $(A \mid \mathbf{v})$. If A is $m \times n$, then $\mathbf{v}$ must be an m-dimensional vector and the augmented matrix will be an $m \times (n + 1)$ matrix-like data structure that has $\mathbf{v}$ as the last column. The reason that I describe this as a data structure is that it is useful for solving a system of linear equations, but it has significance. A matrix is a linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$ and the vector $\mathbf{v}$ is a vector in $\mathbf{F}^m$, but the augmented matrix is simply a tool used to solve a system of m linear equations in n unknowns.
Axiom	A property that describes the relationship between a collection of objects and the operations on those objects. Any truth about that relationship that is derived from those axioms is described as a theorem. Any collection of objects together with defined operations that also satisfies all the axioms automatically also satisfy the corresponding theorems. In this class, we will see axioms for fields (with scalars and scalar addition and scalar multiplication defined), vector spaces on fields (with scalar multiplication and vector addition defined), and linear operators (with scalar multiplication, operator addition, and operator composition (or multiplication) defined).
Backward substitution	An algorithm for finding a solution to a system of linear equations $U\mathbf{u} = \mathbf{v}$ where U is an upper-triangular matrix in row-echelon form. Free variables are associated with any column that does not contain a leading non-zero entry, and then starting with the last row that is not a zero row, we solve for each unknown in terms of the target vector $\mathbf{v}$ and any free variables.
Basis	A basis for a vector space $\mathcal{V}$ or a subspace S is a set of linearly independent vectors such that the span of that set equals the vector space or subspace in question. Given any linearly independent set of vectors, that set is immediately a basis for the span of those vectors.
Best approximation	The best approximation of a vector $\mathbf{v}$ as a linear combination of a given set of n linearly independent vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ is that linear combination that is closest to $\mathbf{v}$. This is equivalent to writing $U = (\mathbf{u}_1 \cdots \mathbf{u}_n)$ and then solving $U^*U\boldsymbol{\alpha} = U^*\mathbf{v}$ for the coefficients $\boldsymbol{\alpha}$. If all the vectors are real, you solve $U^T U\boldsymbol{\alpha} = U^T \mathbf{v}$.
Bilinear function	A bivariate function on vectors is bilinear if $f(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha f(\mathbf{u}, \mathbf{w}) + \beta f(\mathbf{v}, \mathbf{w})$ and $f(\mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}) = \alpha f(\mathbf{u}, \mathbf{v}) + \beta f(\mathbf{u}, \mathbf{w})$. For example, the inner product on real vectors is bilinear, as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$; and the cross product on real 3-dimensional vectors is also bilinear as $(\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w} = \alpha(\mathbf{u} \times \mathbf{w}) + \beta(\mathbf{v} \times \mathbf{w})$ and $\mathbf{u} \times (\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha(\mathbf{u} \times \mathbf{v}) + \beta(\mathbf{u} \times \mathbf{w})$.
Block diagonal	A matrix is block diagonal if comprised of a sequence of square sub-matrices that form the diagonal.
C	The set of all complex numbers $z = \alpha + \beta j$ where α, β
Cauchy–Bunyakovsky–Schwarz inequality	A theorem that says that the inner product $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ of two vectors can never be greater in absolute value than the product of the 2-norms of the vectors: $\ \mathbf{u}_1\ _2 \ \mathbf{u}_2\ _2$. This ensures that $-1 \leq \frac{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}{\ \mathbf{u}_1\ _2 \ \mathbf{u}_2\ _2} \leq 1$. $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \ \mathbf{u}_1\ _2 \ \mathbf{u}_2\ _2$ if and only if each vector

	is a positive scalar multiple of the other, and $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = -\ \mathbf{u}_1\ _2 \ \mathbf{u}_2\ _2$ if and only if each vector is a negative scalar multiple of the other.
Characteristic polynomial	Given an $n \times n$ matrix A , if we calculate the determinant of $\lambda I_n - A$, where λ is a free variable, this defines a polynomial of degree n . This polynomial, by the fundamental theorem of algebra has n roots. Each root is an eigenvalue of the matrix A .
Cholesky decomposition	A symmetric real square matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ that has the additional property that $\langle A\mathbf{u}, \mathbf{u} \rangle > 0$ for each non-zero vector $\mathbf{u} \in \mathbf{R}^n$ may be written in the form LL^T where L is a lower-triangular matrix.
Cofactor expansion	What are you reading? No engineer should ever use cofactor expansions. Period. If you ever implement anything using cofactor expansions, please do not suggest you are in any way associated with Waterloo Engineering.
Collection of vectors	A list of vectors that are in some way enumerated (the first, second, third, etc.) where there may be repetition among the items being listed. These are generally finite in this course.
Column	If the entries of an $m \times n$ matrix A are denoted by $a_{i,j}$, then the k^{th} column are all those entries that have the index $a_{i,k}$. There would be m such entries, and the matrix would have n distinct columns.
Column space	A term not used in this course. It is another description of span of the columns of a matrix A and is therefore equal in every respect to the range of A . This course uses the phrase range of A .
Commutative pair of matrices	A pair of matrices is said to be a commutative pair of matrices if $AB = BA$.
	The property that you can calculate $x + y$ or $y + x$ and get the same result.
Commutativity	Addition and multiplication of scalars in a field are both commutative; vector addition is associative; linear map addition (matrix addition) is also commutative; however, for linear map composition (matrix-matrix multiplication), just because BA is defined, this does not guarantee that AB is defined, and even if AB is defined, in general, $AB \neq BA$. The term commutativity is not used to describe the inner product, which is instead described as being symmetric, and the $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
Complex conjugate	Given a complex number $z = \alpha + \beta j$, the complex conjugate is that complex number $z^* = \alpha - \beta j$. Note that $z^{**} = z$, $zz^* = z ^2$. z is real if and only if $z = z^*$; and z is imaginary if and only if $z = -z^*$.
Complex conjugate pair	Given a non-real complex number z , a complex conjugate pair with that z is itself together with its complex conjugate z, z^* . For example, $3 + 4j$ and $3 - 4j$ form a complex conjugate pair.
Composition	Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, and a second linear map $B: \mathcal{V} \rightarrow \mathcal{W}$, then given a vector $\mathbf{u} \in \mathcal{U}$, it is possible to find $B(A\mathbf{u})$, which must be a vector in $\mathcal{W}$. It can be shown that $B(A\mathbf{u})$ is a linear map from $\mathcal{U}$ to $\mathcal{W}$ and the composition BA is that linear map $BA: \mathcal{U} \rightarrow \mathcal{W}$ such that $(BA)\mathbf{u} = B(A\mathbf{u})$ for all vectors $\mathbf{u} \in \mathcal{U}$. For $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ and $B: \mathbf{F}^m \rightarrow \mathbf{F}^\ell$, the composition $BA: \mathbf{F}^n \rightarrow \mathbf{F}^\ell$ may be found using the algorithm of matrix-matrix multiplication. The composition of the derivative with itself is what we call the second derivative. One could come up with formulas for the second derivative, e.g., the second derivative of $\sin(x)$ is $-\sin(x)$ and the second derivative of x^n is $n(n-1)x^{n-2}$; however, in this case, it is often easier to just calculate the derivative twice.
Conjugate-symmetric matrix	A complex square $n \times n$ matrix is conjugate symmetric if $a_{i,j} = a_{j,i}^*$ for all column and row indices or $A = A^*$ where A^* is the conjugate transpose of A . This means that all diagonal entries must be real. A complex conjugate-symmetric $n \times n$ matrix has n real eigenvalues when counting multiplicity.

Conjugate transpose	The conjugate transpose of a $m \times n$ matrix $A: \mathbf{C}^n \rightarrow \mathbf{C}^m$ is an $n \times m$ matrix $A^*: \mathbf{C}^m \rightarrow \mathbf{C}^n$ where the $(i,j)^{\text{th}}$ entry of the conjugate transpose equals $a_{j,i}^*$. The conjugate transpose is that matrix such that $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^*\mathbf{v} \rangle$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.
Cramer's rule	A horribly inefficient means of solving a system of n linear equations in n unknowns assuming there is a unique solution. It may be okay for solving a system of two equations in two unknowns or three equations in three unknowns, but otherwise it is absolutely off limits. It is an algorithm, and it is not entirely obvious why it is correct.
Cross product	Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ that are not parallel, the cross product returns a vector $\mathbf{u} \times \mathbf{v}$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ is parallel to $\mathbf{v}$, the cross product is $\mathbf{0}_3$. The cross product is bilinear and $\ \mathbf{u} \times \mathbf{v}\ _2 = \ \mathbf{u}\ _2 \ \mathbf{v}\ _2 \sin(\theta)$ where θ is the angle between the two vectors. The cross product is peculiar to $\mathbf{R}^3$ and is a means of a vector that is orthogonal to two given vectors in that space. It is not a product in the sense of scalar multiplication, as there is no identity vector, it is not commutative ($\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$), it is not associative ($\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$); however, it does distribute over vector addition: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$. It is peculiar in that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ for all vectors $\mathbf{u}$.
Decomposition	The process of writing a more complex object in terms of easier objects. A prime decomposition of an integer writes an integer as a product of prime numbers; e.g., $2020 = 2^2 \times 5 \times 101$. A matrix decomposition is writing an arbitrary matrix A as a composition of simpler matrices, so $A = BC$ or $A = BCD$ where B , C and possibly D are simpler matrices with prescribed characteristics; e.g., upper triangular matrix, lower triangular matrix, permutation matrix, diagonal matrix, orthogonal matrix, etc.
Dependent variable	A variable in a solution to a linear equation or system of linear equations, the value of which is either fixed or depends on the values given to any free or independent variables.
Determinant	The determinant of a square matrix $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ stores the ratio of the change in volume after a region is mapped under A , so that if R has a volume of $\text{vol}(R)$ in $\mathbf{F}^n$ and AR represents each point in R mapped by A , then $\text{vol}(AR) = \det(A) \text{vol}(R)$. The sign of $\det(A)$ indicates whether orientation is preserved or if it is flipped (a mirror reflection results). $\det(A)$ is non-zero if and only if A is invertible. If A is an upper- or lower-triangular matrix, then $\det(A)$ is the product of the diagonal entries. DO NOT use the determinant to check if a matrix is invertible, because if a matrix is not invertible, numerical error may result in a calculation of a determinant that is non-zero; however, even if the determinant is very small, the matrix may still be perfectly invertible. For example, $\det(\varepsilon I_n) = \varepsilon^n$ for $\varepsilon \neq 0$ can be made arbitrarily small, but $(\varepsilon I_n)^{-1} = \varepsilon^{-1} I_n$, so it is clearly and easily invertible.
Diagonal entry	A diagonal entry of a matrix is any entry $a_{i,i}$; that is, where the row and column indices are equal. This is contrasted with off-diagonal entries.
Diagonal matrix	An $m \times n$ matrix A is said to be diagonal if all entries $a_{i,j}$ where $i \neq j$ equals zero. A zero matrix is also a diagonal matrix. A diagonal matrix is invertible so long as all the diagonal entries are non-zero. The inverse of a diagonal matrix D is that diagonal matrix that has entries $d_{i,i}^{-1}$ on the diagonal.
Dimension	Given a vector space (be it a vector space or a subspace of a vector space), the dimension of that vector space is the number of vectors in any basis for that vector space. If the number of vectors is finite, then all basis must have the same number of vectors in it; otherwise, we say the dimension of the vector space is infinite. The dimension of a subspace must be less than or equal to the dimension of the vector space containing it. The dimension of $\mathbf{F}^n$ is n . The dimension of all polynomials of degree less than or equal to n with coefficients in $\mathbf{F}$ is $n + 1$. The dimension of the vector space of all polynomials is infinite.
Distance between vectors	The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ given a norm is defined as $\ \mathbf{u} - \mathbf{v}\ $. The distance is symmetric, so $\ \mathbf{u} - \mathbf{v}\ = \ \mathbf{v} - \mathbf{u}\ $.
Distributivity	Given two operations, the first usually analogous to multiplication, and the second usually analogous to addition, the property $x(y + z) = xy + xz$ holds for all x, y and z .

	In all fields, multiplication distributes over addition. Scalar multiplication distributes over vector addition in vector spaces. Scalar multiplication distributes over linear map addition. Linear map composition distributes over linear map addition.
Domain	Given a function f , that function maps entries in the domain to values in the co-domain. Generally, a real-valued function of a real variable is described as $f: \mathbf{R} \rightarrow \mathbf{R}$ where the first $\mathbf{R}$ is the domain, and a linear map A is described as $A: \mathcal{U} \rightarrow \mathcal{V}$ where the vector space $\mathcal{U}$ is the domain.
<i>Dot product</i>	Another name for the inner product. This is a term often used in vector calculus. It is not used in this course.
Eigenspace	If λ is an eigenvalue of a linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$, then the eigenspace of λ is all vectors in $\mathcal{U}$ such that $A\mathbf{u} = \lambda\mathbf{u}$. The collection of all such vectors $\mathbf{u} \in \mathcal{U}$ forms a subspace of $\mathcal{U}$. If λ is an eigenvalue, the dimension of this subspace is at least one, and if λ is not an eigenvalue, then the dimension of this subspace is zero.
Eigenvalue	A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ has an eigenvalue λ if there is a vector $\mathbf{u}$ such that $A\mathbf{u} = \lambda\mathbf{u}$. In this case, $\mathbf{u}$ is described as an eigenvector corresponding to the eigenvalue λ . If $A: \mathbf{C}^n \rightarrow \mathbf{C}^n$, then A has at least one eigenvalue. If $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, it may not even have one real eigenvalue, although if n is odd, A must have at least one real eigenvalue.
Eigenvalue decomposition	A real symmetric square matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ can be written as $A = UDU^T$ where U is an orthogonal matrix comprised of n orthogonal eigenvectors, and D is a diagonal matrix where $\mathbf{u}_k$ (the k^{th} column of U) is a normalized eigenvector corresponding to the eigenvalue $d_{k,k}$. There are other matrices for which $A = UDU^{-1}$ but where the eigenvectors are not orthogonal and therefore the inverse must be explicitly calculated.
Eigenvector	If for a linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$, there is a non-zero vector $\mathbf{u}$ such that $A\mathbf{u} = \lambda\mathbf{u}$, then $\mathbf{u}$ is said to be an eigenvector of the matrix A and it is an eigenvector corresponding to the eigenvalue λ . If $\mathbf{u}$ is an eigenvector corresponding to an eigenvalue λ , then so is every other non-zero multiple of $\mathbf{u}$.
$\mathbf{e}_k$	A canonical basis vector in $\mathbf{F}^n$ where the vector is n -dimensional and all entries are zero except for the k^{th} entry, which is 1.
<i>Euclidean norm</i>	Another phrase for the 2-norm of a vector. This phrase is not used in this course.
Even function	A function that satisfies $f(-x) = f(x)$. This is used in demonstrating that every function is the sum of an odd and an even function, an observation contrasted with every operator being the sum of a self-adjoint matrix and skew-adjoint matrix. See odd functions.
<i>Even permutation matrix</i>	A permutation matrix that has a determinant equal to 1, meaning that orientation is preserved. This phrase is not used in this course. See odd permutation matrices.
<i>Even polynomial</i>	A polynomial $p(x)$ that has only even powers of x . Every even polynomial is an even function. See odd polynomials.
$\mathbf{F}$ and $\mathbf{F}^n$	When a result is independent of the scalars being either real or complex, we use $\mathbf{F}$ to represent a field. Thus, this indicates we may use either $\mathbf{R}$ or $\mathbf{C}$, or $\mathbf{R}^n$ or $\mathbf{C}^n$, respectively.
<i>Factorization</i>	Another term for decomposition. In this course, we only ever refer to matrix decomposition; however, in the literature, you may see the phrase matrix factorization.
Field	A collection of scalars that behaves like the rational numbers: you can add and multiply them, there is a 0 and a 1, the operations are commutative and associative, multiplication distributes over addition, and there are additive inverses for all scalars and multiplicative inverses for all scalars other than zero. The fields we see in this class include the rationals, the reals, and the complex numbers. The irrational numbers are not a field, for you can add two irrational numbers and get something that is not an irrational number.
Floating-point operation	An arithmetic operation where the operands are floating-point numbers (float or double). This is usually abbreviated as a FLOP.
FLOP	See floating-point operation.

Free variable	A variable that may take on any real or complex number. Also known as an independent variable.
Fundamental theorem of algebra	Every polynomial of degree n with complex coefficients has n complex roots when counting multiplicity.
Gaussian elimination	A process by which a variable in the k^{th} equation is eliminated from all subsequent equations by adding appropriate multiples of the k^{th} equation to eliminate that variable from all subsequent equations. We start with the first variable in any equation, and proceed on to subsequent variables. We can only eliminate the leading non-zero variable from all subsequent equations if all equations starting with the k^{th} have the variables up to that leading non-zero variable already set to zero by previous eliminations. The result will be a matrix or augmented matrix that is in row-echelon form.
Gram-Schmidt algorithm	An algorithm for taking a collection of vectors and converting them into a set of orthonormal vectors such that the span of the collection equals the span of the set of orthonormal vectors. The orthonormal vectors are a basis for that span. If the original collection was not linearly independent, some of the resulting vectors will be the zero vector, which should be discarded.
<i>Hermitian symmetric matrix</i>	Another phrase describing a matrix that is conjugate symmetric. This term is not used in this course.
<i>Hermitian transpose</i>	Another phrase describing the conjugate transpose. This term is not used in this course.
Homogeneous linear equation	A linear equation where the constant coefficient is zero; that is, an equation in n unknowns $x_1, \dots, x_n$ where the equation is of the form $a_1x_1 + \dots + a_nx_n = 0$ where the a_i s are given and are either all real or all complex numbers. Every homogeneous linear equation is also a linear equation.
i	In mathematics, the usual representation of the imaginary unit. See j . i is not used in engineering, as i is generally used to represent current and complex numbers are integral to the analysis of alternating current (AC).
$\hat{\mathbf{i}}$	The first canonical basis vector in $\mathbf{R}^3$ or $\mathbf{C}^3$: $\hat{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
I_n	The representation in this course of the $n \times n$ identity matrix.
Identity matrix	The identity matrix that maps $\mathbf{R}^n$ or $\mathbf{R}^n$ or $\mathbf{C}^n$ or $\mathbf{C}^n$ is the $n \times n$ matrix with all diagonal entries equal to 1 and all other entries equal to zero and is represented as I_n .
Identity operator	The identity operator is that linear operator $I: \mathcal{U} \rightarrow \mathcal{U}$ where $I\mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u} \in \mathcal{U}$. If the vectors space is either $\mathbf{R}^n$ or $\mathbf{C}^n$, the identity operator is the identity matrix which is represented as I_n .
Image	Given a linear map A and a vector $\mathbf{u}$, the image of $\mathbf{u}$ under A is the vector $A\mathbf{u}$. This is unique. Every vector in the domain has an image.
Imaginary component	Given a complex number $z = \alpha + \beta j$, the imaginary component is the real number β and it is often written that $\Im(z) = \beta$.
<i>Imaginary part</i>	Another name for the imaginary component. This term is not used in this course.
Imaginary unit	A symbol representing the square root of -1 . In engineering, this is usually j and $j^2 = -1$.
Independent variable	Another term for a free variable.
Infinity-norm or ∞-norm	For finite n -dimensional vectors, the infinity-norm of a real or complex vector is $\ \mathbf{u}\ _\infty = \max_{k=1, \dots, n} \{ u_k \}$.
Inner product	Any function that maps each pair of real vectors to a real number, or each pair of complex vectors to a complex number. For real vectors, the inner product must satisfy $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$, $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$, and $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$ for any real number α . For complex vectors we require

$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and therefore $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{v} \rangle$. For two real n -dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, we define $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k v_k$ and for two complex n -dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, we define $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k^* v_k$.

Inverse of a square matrix	A matrix $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ has an inverse if there is a second $n \times n$ $A^{-1}: \mathbf{F}^n \rightarrow \mathbf{F}^n$ that has the property if $A\mathbf{u}_1 = \mathbf{u}_2$, then $A^{-1}\mathbf{u}_2 = \mathbf{u}_1$. Also, the inverse is the only linear operator such that $AA^{-1} = I_n = A^{-1}A$.
Inverse of a linear operator	A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ has an inverse if there is a second linear operator $A^{-1}: \mathcal{U} \rightarrow \mathcal{U}$ that has the property if $A\mathbf{u}_1 = \mathbf{u}_2$, then $A^{-1}\mathbf{u}_2 = \mathbf{u}_1$. Also, the inverse is the only linear operator such that $AA^{-1} = I = A^{-1}A$.
Invertible matrix	A matrix for which an inverse exists.
Isometric operator	A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ is isometric if $\ A\mathbf{u}\ _2 = \ \mathbf{u}\ _2$ for all vectors $\mathbf{u} \in \mathcal{U}$. A linear operator is isometric if and only if $A^{-1} = A^*$ where A^* is the adjoint, and if A is an $n \times n$ isometric real matrix, then A is described as <i>orthogonal</i> and if A is an $n \times n$ isometric complex matrix, then A is described as <i>unitary</i> .
j	The usual representation of the imaginary unit in engineering.
$\hat{\mathbf{j}}$	The second canonical basis vector in $\mathbf{R}^3$ or $\mathbf{C}^3$: $\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
$\hat{\mathbf{k}}$	The third canonical basis vector in $\mathbf{R}^3$ or $\mathbf{C}^3$: $\hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
<i>Kernel</i>	Another name for the null space of a linear map. This term is not used in this course.
Leading non-zero entry	If a row in a matrix is not a zero row, then that row must have a first entry that has a non-zero value. That entry is the leading non-zero entry of that row.
Leading zeros	A leading zero is any zero that appears in a row to the left of the first non-zero entry of that row. If the first entry of the row is a non-zero value, then the row has no leading zeros. A row in a matrix that is not a zero row has k leading zeros if the first k entries in that row are zero and the $(k+1)^{\text{st}}$ is a non-zero entry. That $(k+1)^{\text{st}}$ entry is the leading non-zero entry of that row.
<i>Length of a vector</i>	Another description for the norm of a vector. This phrase is not used in this course.
Linear equation	An equation in n unknowns $x_1, \dots, x_n$ where the equation is of the form $a_1x_1 + \dots + a_nx_n = b$ where the a_i s and the b are given and are either all real or all complex numbers. b is called the constant coefficient.
Linear map	A linear map is a function A that maps vectors in a vector space into another vector space, so $A: \mathcal{U} \rightarrow \mathcal{V}$. The properties of a linear map are that $A(\alpha \mathbf{u}) = \alpha A\mathbf{u}$ and $A(\mathbf{u}_1 + \mathbf{u}_2) = A\mathbf{u}_1 + A\mathbf{u}_2$. Note that functions are generally written as $f(x)$ or $\sin(x)$, but with a linear map, if there is only one vector as the argument, we prefer to write $A\mathbf{u}$ instead of $A(\mathbf{u})$. This is because when a linear map is a matrix, $A\mathbf{u}$ is interpreted as matrix-vector multiplication. Also known as a linear transformation.
Linear operator	A linear operator A is a linear map that maps vectors in a vector space into itself, so $A: \mathcal{U} \rightarrow \mathcal{U}$; that is, the domain and co-domain are the same vector space.
Linearly dependent collection of vectors	A collection of vectors where at least one vector can be written as a linear combination of other vectors in the collection. A set $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is linearly dependent if and only if there is more than the trivial solution to $\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n = \mathbf{0}$, and if there is more than one, there are infinitely many solutions. A collection that has a vector repeated is automatically linearly dependent. A collection that contains the zero vector is automatically linearly dependent, even if the zero vector is the only vector in the collection.
Linearly independent set of vectors	A set of vectors where no vector can be written as a linear combination of the other vectors in the set. A set $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is linearly independent if and only if the only

	solution to $\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$ is when $\alpha_1 = \cdots = \alpha_n = 0$. Any collection of vectors that has either the zero vector $\mathbf{0}$ or duplicate entries cannot be linearly independent.
Linear transformation	Another name for linear maps. This is not a term used in this course.
Lower-triangular matrix	A matrix is said to be lower triangular if each entry $a_{i,j}$ where $i < j$ is zero. That is, each entry that is to the right or above a diagonal entry must be zero. The zero matrix and diagonal matrices are lower-triangular matrices.
LU decomposition	See PLU decomposition
Magnitude of a complex number	Another phrase for the absolute value of a complex number. This phrase is not used in this course.
Magnitude of a vector	Another phrase for the norm of a vector. This phrase is not used in this course.
Matrix-matrix multiplication	An operation so defined so that $(BA)\mathbf{u}$ produces the exact same output as $B(A\mathbf{u})$ for all possible inputs $\mathbf{u}$. If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ and $B: \mathbf{F}^m \rightarrow \mathbf{F}^\ell$, then $BA: \mathbf{F}^n \rightarrow \mathbf{F}^\ell$. The most obvious means of remembering such a product to recall that A is a matrix of n columns, so $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$. If you remember the definition of matrix-vector multiplication, then $BA = (B\mathbf{a}_1 \cdots B\mathbf{a}_n)$; that is, multiply each column of A by B . Also called matrix multiplication.
Matrix-vector multiplication	If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ and $\mathbf{u}$ is an n -dimensional vector, the product $A\mathbf{u}$ is an m -dimensional vector $\mathbf{v}$ and $\mathbf{v}$ is a linear combination of the n column vectors of A where the coefficients are corresponding entries of $\mathbf{u}$. Thus, if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, then $\mathbf{v} = u_1 \mathbf{a}_1 + \cdots + u_n \mathbf{a}_n$. Alternatively, you can also use $v_i = \sum_{j=1}^n a_{i,j} u_j$.
Minors	See cofactor expansion. Do not use minors, ever.
Monotonic increasing	A real-valued function of a real variable defined on a finite open or closed interval (e.g., (a, b) or $[a, b]$), a semi-infinite open or closed interval (e.g., (a, ∞) or $(-\infty, b]$), or all of $\mathbf{R}$, is said to be monotonic increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ for x_1 and x_2 on that domain. For example, x^2 on $[0, \infty)$, $\tan(x)$ on $(-\pi/2, \pi/2)$, $\ln(x)$ on $(0, \infty)$ and e^x on $\mathbf{R}$. A constant function is also monotonic increasing. See strictly monotonic increasing.
Multiplicative identity	In a field, a 1 such that $\alpha \cdot 1 = \alpha$ for all scalars α in the field, and every non-zero scalar α has a multiplicative inverse denoted α^{-1} or $1/\alpha$ such that $\alpha \cdot \alpha^{-1} = \alpha \cdot (1/\alpha) = 1$.
Multiplicative inverse	In a field, each non-zero scalar α has a multiplicative inverse α^{-1} or $(1/\alpha)$ such that $\alpha \cdot \alpha^{-1} = \alpha \cdot (1/\alpha) = 1$. For a rational number m/n , the multiplicative inverse is n/m , for a real number, one must usually perform long division to get an approximation of the multiplicative inverse, and the multiplicative inverse of $z = \alpha + \beta j$ is $z^*/ z ^2$.
Multiplicity of a root	A polynomial of degree n has a root r of multiplicity k if in the vicinity of r , the function <i>looks</i> like $(x - r)^k$. Equivalently, if we divide the polynomial through by $(x - r)$, the result is a polynomial of degree $n - 1$ with zero remainder and that polynomial has a root of r of multiplicity $k - 1$. Saying that a polynomial has a root of multiplicity zero at r is equivalent to saying that polynomial does not have a root at r .
Non-invertible matrix	A square $n \times n$ matrix for which no inverse exists. A matrix is not invertible or non-invertible if and only if the null space is more than just the zero vector $\mathbf{0}_n$ which is equivalent to (for square matrices) the range is not all of $\mathbf{F}^n$. Also known as a singular matrix.
Non-singular matrix	A matrix that is invertible. This term is not used in this course.
Norm	A means of measuring the length of a vector. The default and most common norm is the square root of the inner product, called the 2-norm: $\ \mathbf{u}\ _2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$. This is the norm induced by the inner product. There are two other common norms: the 1-norm and the infinity-norm. All norms have three properties: $\ \mathbf{u}\ \geq 0$ and $\ \mathbf{u}\ = 0$ if and only if $\mathbf{u} = \mathbf{0}$, $\ \alpha \mathbf{u}\ = \alpha \ \mathbf{u}\ $, and $\ \mathbf{u} + \mathbf{v}\ \leq \ \mathbf{u}\ + \ \mathbf{v}\ $.
Normalization	The process of taking a non-zero vector $\mathbf{u}$ and dividing it by its 2-norm to make it a unit vector.
Normal matrix	A matrix that is a normal operator. If $A: \mathbf{C}^n \rightarrow \mathbf{C}^n$ and A is normal, then A is diagonalizable with a unitary matrix, so $A = UDU^*$. A $A: \mathbf{C}^n \rightarrow \mathbf{C}^n$ and A is normal, it

	is only diagonalizable with an orthogonal (real) matrix if A is symmetric. If it is not symmetric, it is still diagonalizable as a complex matrix, but U and D will have complex entries, even if UDU^* equals the real matrix.
Normal operator	A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ is normal if it commutes with its adjoint; that is, $AA^* = A^*A$.
Null space of a linear map	Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, the null space is all vectors that are mapped to the zero vector in the co-domain; that is, all vectors $\mathbf{u}$ such that $A\mathbf{u} = \mathbf{0}_{\mathcal{V}}$. The null space is a subspace of the domain. A linear map is one-to-one if and only if the null space is $\{\mathbf{0}_{\mathcal{U}}\}$.
Nullity	The nullity of a linear operator A is the dimension of the null space. This term is not used in this course.
Odd function	A function that satisfies $f(-x) = -f(x)$. This is used in demonstrating that every function is the sum of an odd and an even function, an observation contrasted with every operator being the sum of a self-adjoint matrix and skew-adjoint matrix.. See even functions.
Odd permutation matrix	A permutation matrix that has a determinant equal to -1 , meaning that orientation is flipped or reflected. This phrase is not used in this course. See even permutation matrices.
Odd polynomial	A polynomial $p(x)$ that has only odd powers of x . Also used to describe polynomials with bizarre personalities. Every odd polynomial is an odd function. See even polynomials.
Off-diagonal entry	An off-diagonal entry of a matrix is any entry a_{ij} ; where $i \neq j$; that is, any entry where the row and column indices differ. This is contrasted with diagonal entries.
1-norm	For finite n -dimensional vectors, the one-norm of a real or complex vector is $\ \mathbf{u}\ _1 = \sum_{k=1}^n u_k $.
One-to-one	A linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ is one-to-one if every $\mathbf{u} \in \mathcal{U}$ is mapped to a unique vector in $\mathcal{V}$. That is equivalent to saying that if $A\mathbf{u}_1 = A\mathbf{u}_2$, then $\mathbf{u}_1 = \mathbf{u}_2$. This is also equivalent to saying the dimension of the domain is equal to the dimension of the range.
One-to-one and onto	A linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ is one-to-one and onto if every $\mathbf{u} \in \mathcal{U}$ is mapped to a unique vector in $\mathcal{V}$ and the range is all of $\mathcal{V}$. A one-to-one and onto map $A: \mathbf{R}^m \rightarrow \mathbf{R}^n$ must be an operator, so the domain and co-domain must be the same. A linear operator on a finite-dimensional vector space is invertible if and only if it is one-to-one and onto.
Onto	A linear map $A: \mathcal{U} \rightarrow \mathcal{V}$ is onto if every $\mathbf{v} \in \mathcal{V}$ is the image of at least one vector in the domain $\mathcal{U}$. That is equivalent to saying that if $\mathbf{v} \in \mathcal{V}$, then there must exist a $\mathbf{u} \in \mathcal{U}$ such that $A\mathbf{u} = \mathbf{v}$.
Orientation	Orientation describes a relationship between <i>left</i> and <i>right</i> . Most intuitively, you may hold an object in your left hand, but if you look in a mirror, it appears as if you are holding that object in your right hand. This is because the mirror acts as the row operation of swapping two entries in a matrix. Swapping two entries of a matrix will always result in a flip in orientation. More generally, if $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ and $\det(A) > 0$, then after mapping all vectors under A , orientation is always preserved, whereas if $\det(A) < 0$, orientation is flipped.
Origin	Another term for the zero vector, especially in $\mathbf{R}^n$, and when a vector is being described as being relative to the zero vector.
Orthogonal vectors	Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if the inner product is zero. This is equivalent to saying that the angle between them is 90° as well as saying they are perpendicular to each other. This also says that one vector has no information about the other, meaning that the best approximation of $\mathbf{v}$ as a multiple of $\mathbf{u}$ is $0\mathbf{u} = \mathbf{0}$. The zero vector is orthogonal to all vectors. A set of vectors is orthogonal if the inner product between any two different vectors in the collection is zero.
Orthogonal matrix	A square real matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is said to be <i>orthogonal</i> if $\ A\mathbf{u}\ _2 = \ \mathbf{u}\ _2$ for all vectors $\mathbf{u} \in \mathbf{R}^n$. A square matrix is orthogonal if and only if the columns form an orthonormal basis for $\mathbf{R}^n$, which is true if and only if the rows, too, form an

	orthonormal basis for $\mathbf{R}^n$. A real square matrix is orthogonal if and only if $A^{-1} = A^T$.
Orthonormal basis	Given a vector space, an orthonormal basis is a set of vectors such that the span of the vectors is the given vector space, the vectors are all unit vectors, and the vectors are mutually orthogonal.
Orthonormal vectors	Two vectors are orthonormal if they are orthogonal vectors and their 2-norm is one. A set of vectors is orthonormal if the inner product of two vectors in the set is either 0 if the vectors are different or 1 if they are equal.
Overdetermined system of linear equations	Any system of linear equations where the number of equations is greater than the number of unknowns. There can be either no solutions, exactly one unique solution or infinitely many solutions; however, if you were to randomly generate an overdetermined system of linear equations, there will be, in general, no solutions.
Permutation matrix	A permutation matrix is a linear operator $P: \mathbf{F}^m \rightarrow \mathbf{F}^m$ such that $P\mathbf{u}$ permutes (interchanges) the entries of $\mathbf{u}$ and PA permutes (interchanges) the rows of A where A is an $m \times n$ matrix. All entries are zero except that each row has exactly one 1 and each column has exactly one 1. The identity matrix is a permutation matrix. Each permutation matrix P is invertible and $P^{-1} = P^T$ for real matrices and P^* for complex matrices.
Perpendicular vectors	Another word for orthogonal.
Phase of a complex number	Another name for the angle of a complex number.
PLU decomposition	What this course calls an LU decomposition. Every matrix $A: \mathbf{F}^m \rightarrow \mathbf{F}^n$ can be written as a product of three matrices $A = PLU$ where P is an $m \times m$ permutation matrix, L is a lower-triangular matrix with 1s on the diagonal, and U is an upper-triangular matrix in row-echelon form.
Polar representation	Writing a complex number as its absolute value and its angle θ. It is often written as $z \angle \theta$ or $z \angle \theta$ and this is read as “the absolute value of zed phase θeta.” It can also be written as $z e^{j\theta}$. This is contrasted with the rectangular representation.
Pre-image	Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, if $\mathbf{v} \in \mathcal{V}$, then any vector in $\mathbf{u} \in \mathcal{U}$ such that $A\mathbf{u} = \mathbf{v}$ is called a pre-image of $\mathbf{v}$. When we solve $A\mathbf{u} = \mathbf{v}$ for $\mathbf{u}$, we are searching for all pre-images of $\mathbf{v}$. It may happen that a vector $\mathbf{v}$ has no pre-image.
Principal minors	See cofactor expansions. Do not use principal minors, ever.
Projection	The projection of one vector $\mathbf{u}$ onto another vector $\mathbf{v}$ is that scalar multiple of $\mathbf{v}$ that best approximates $\mathbf{u}$, meaning that it is that scalar multiple of $\mathbf{v}$ that is closest to $\mathbf{u}$. It is written as $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$. In the special case where $\mathbf{v}$ is a unit vector, this calculation can be simplified to $\text{proj}_{\hat{\mathbf{v}}}(\mathbf{u}) = \langle \hat{\mathbf{v}}, \mathbf{u} \rangle \hat{\mathbf{v}}$.
Pythagorean theorem	A theorem that says if you are given a right-angled triangle, then the square of the length of the hypotenuse equals the sum of the squares of the length of the other sides. For vectors, this means $\ \mathbf{u} + \mathbf{v}\ _2^2 = \ \mathbf{u}\ _2^2 + \ \mathbf{v}\ _2^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
QR decomposition	Every matrix A can be written in the form of QR where Q is an isometric matrix (orthogonal matrix for real matrices, unitary for complex matrices) and where R is an upper-triangular (or “right”-triangular) matrix. The term is used, and when it is used, it is defined, but we do not cover this decomposition in this course.
R	The set of all real numbers. This does not include $\pm\infty$.
Range of a linear map	Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, the range is all vectors that are mapped to in the co-domain by at least one vector in the domain; that is, all vectors $A\mathbf{u}$ such that $\mathbf{u} \in \mathcal{U}$. The range is a subspace of the co-domain. A linear map is onto if and only if the range equals the co-domain.

Rank	The rank of the matrix A is the number of non-zero rows of a matrix that is row-equivalent to A while also in row-echelon form. It is equal to the dimension of the range. If the rank equals the dimension of the co-domain, the matrix is onto.
Real component	Given a complex number $z = \alpha + \beta j$, the real component is the real number α and it is often written that $\Re(z) = \alpha$.
<i>Real part</i>	Another name for the real component. This term is not used in this course.
Real vector	A vector, the entries of which are real numbers. If the vector is a function, the function must map the domain onto the real numbers.
Rectangular representation	Writing a complex number as its real component plus j multiplied by its imaginary component, or $z = \alpha + \beta j$. This is contrasted with the polar representation.
Reflection	In $\mathbf{F}^n$, the process of reflecting each point through an $(n - 1)$ -dimensional plane whereby each point $\mathbf{u}$ is moved as follows: if $\mathbf{v}$ is the point on the plane closest to $\mathbf{u}$, then $\mathbf{u}$ is reflected to $\mathbf{u} + 2(\mathbf{v} - \mathbf{u}) = 2\mathbf{v} - \mathbf{u}$. The row operation of swapping Row i and Row j is equivalent to reflecting each point through the $(n - 1)$ -dimensional subspace (also being a plane) where $v_i = v_j$. We will generally not require this interpretation in this course.
Right-hand rule	Given two vectors $\mathbf{u}$ and $\mathbf{v}$, the direction of the cross product can be determined by pointing the index finger straight out in the direction of $\mathbf{u}$, the middle finger in the direction of $\mathbf{v}$ (with that finger necessarily being pointed somewhat in), in which case the direction of the cross product $\mathbf{u} \times \mathbf{v}$ will be in the direction of the thumb.
Root of unity	A complex number z is an n^{th} root of unity if $z^n = 1$. For example, there are four complex numbers that are 4^{th} -roots of unity: $1, j, -1$ and $-j$.
Row	If the entries of an $m \times n$ matrix A are denoted by a_{ij} , then the k^{th} row are all those entries that have the index a_{kj} . There would be n such entries, and the matrix would have m distinct rows.
Row equivalent	Two matrices A and B are row-equivalent if there is a sequence of row operations that transforms one matrix to the other. Row equivalency has the properties that each matrix is row equivalent to itself; A is row equivalent to B if and only if B is row equivalent to A ; and if A is row equivalent to B and B is row equivalent to C , then A is row equivalent to C .
Row operation	An operation that works on entire rows of a matrix. The three row operations we use are multiplying a row by a non-zero scalar, swapping two rows, and adding a multiple of one row onto another. All of these operations can be performed as a result of a matrix-matrix multiplication. All of these operations can be undone or inverted. These row operations can be interpreted as scaling in one dimension, reflecting in an $(n - 1)$ -dimensional subspace, and shearing, respectively; however, we will generally not require these interpretations in this course.
Row-echelon form	A matrix (adjacency or otherwise) is said to be in row-echelon form if each subsequent row has more leading zeros than the previous. Consequently, all zero-rows, if any, must be at the bottom of the matrix.
Reduced row echelon form	An unnecessary form, and therefore ignored in this course, whereby after a matrix is transformed into one that is in row-echelon form, each non-zero row is subsequently divided by the value of the leading non-zero entry, thus placing ones in each of these locations. Implementing this would be a waste of computation and thus this form is ignored.
Scalar	A value in a field, either $\mathbf{R}$ or $\mathbf{C}$, and if the field doesn't matter, we use $\mathbf{F}$.
Scalar multiplication	Given a collection of vectors associated with a field $\mathbf{F}$, scalar multiplication is an operation $\alpha \mathbf{u}$ where this produces a new vector and where this operation distributes over vector addition: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$. It is always true that $1 \mathbf{u} = \mathbf{u}$ and $0 \mathbf{u} = \mathbf{0}$ where $\mathbf{0}$ is the zero vector.
Scaling in one dimension	In $\mathbf{F}^n$, given a vector $\mathbf{v}$, scaling a point $\mathbf{u}$ by α in the direction of $\mathbf{v}$ is equivalent to replacing $\mathbf{u}$ by $\mathbf{u} + (\alpha - 1)\text{proj}_{\mathbf{v}}(\mathbf{u})$. The row operation of multiplying Row i by α is equivalent to scaling in the direction of $\hat{\mathbf{e}}_i$, the i^{th} canonical basis vector. We will generally not require this interpretation in this course.

Set of vectors	A collection of vectors but where repetition is not allowed.
Sesquilinearity	A bivariate function on complex vectors is sesquilinear if $f(\mathbf{u}, \alpha \mathbf{v}) = \alpha f(\mathbf{u}, \mathbf{v})$ but $f(\alpha \mathbf{u}, \mathbf{v}) = \alpha^* f(\mathbf{u}, \mathbf{v})$. For example, the inner product on complex vectors is sesquilinear, as $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$ but $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{v} \rangle$. <i>Sesqui</i> means “one and a half times” in Latin.
Shear	In $\mathbf{F}^n$, given non-parallel vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, a shear moves a point $\mathbf{u}$ to a new point $\mathbf{u} + \alpha \mathbf{v}_1$ where α is proportional to the inner product of $\mathbf{u}$ and $\mathbf{v}_2$. The row operation of adding α times Row i onto Row j is equivalent to moving $\mathbf{u} + \alpha \langle \hat{\mathbf{e}}_i, \mathbf{u} \rangle \hat{\mathbf{e}}_j$ where $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ are the i^{th} and j^{th} canonical basis vectors, respectively. We will generally not require this interpretation in this course.
Singular matrix	Another name for a non-invertible matrix. A matrix that is invertible is also described as being non-singular.
Singular value	A singular value of a linear map A is the square root of a non-zero eigenvalue of the map AA^* or A^*A ; where if there are less than $\min\{m, n\}$ non-zero singular values, the balance are zero. Recall that all AA^* and A^*A are both symmetric and have all non-negative eigenvalues (all eigenvalues are greater-than or equal-to zero). If A is a real matrix, A^* is the transpose, and if A is a complex matrix, A^* is the conjugate transpose. The singular values of a symmetric or normal matrix A are the absolute values of the eigenvalues of the matrix A .
Singular-value decomposition	Abbreviated as SVD. An $m \times n$ matrix A may be written as $A = USV^*$ where U is an $m \times m$ isometric matrix, V is an $n \times n$ isometric matrix, and S is an $m \times n$ diagonal matrix. The entries of S are the singular values of A . The columns of U are the eigenvectors corresponding to the eigenvalues of AA^* and the columns of V are the eigenvectors of A^*A . If A is a real matrix, A^* is the transpose, and if A is a complex matrix, A^* is the conjugate transpose.
Skew-symmetric matrix	A real square $n \times n$ matrix is skew symmetric if $a_{i,j} = -a_{j,i}$ for all column and row indices. All diagonal entries must therefore be zero. A real skew-symmetric $n \times n$ matrix has n imaginary eigenvalues when counting multiplicity.
Span	The span of a collection of vectors is all linear combinations of those vectors. The span is always a subspace of the vector space from which the vectors are taken.
<i>Standard basis</i>	Another phrase describing a canonical basis. This phrase is not used in this class.
Strictly monotonic increasing	A real-valued function of a real variable defined on a finite open or closed interval (e.g., (a, b) or $[a, b]$), a semi-infinite open or closed interval (e.g., (a, ∞) or $(-\infty, b]$), or all of $\mathbf{R}$, is said to be strictly monotonic increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for x_1 and x_2 on that domain. For example, x^2 on $[0, \infty)$, $\tan(x)$ on $(-\pi/2, \pi/2)$, $\ln(x)$ on $(0, \infty)$ and e^x on $\mathbf{R}$. A strictly monotonic increasing function must be one-to-one on the domain. See monotonic increasing.
Sub-diagonal entry	A sub-diagonal entry of a matrix is any entry $a_{i,i-1}$; where $i > 1$ and i is a valid row index in the matrix. This is contrasted with super-diagonal entries.
Subspace	Given a vector space, a subspace is a subset of that vector space such that if you add two vectors in the subspace together, the sum must still be in the subspace, and if you multiply any vector in the subspace by a scalar, that scalar multiple must still be in the subspace. A subspace is a vector space. The intersection of two subspaces is a subspace. The span of any collection of vectors is a subspace. The entire vector space and the subset containing only the zero vector $\{\mathbf{0}\}$ are both subspaces of a vector space.
Super-diagonal entry	A sub-diagonal entry of a matrix is any entry $a_{i,i+1}$ where $i+1$ is a valid column index in the matrix. This is contrasted with sub-diagonal entries.
SVD	See singular-value decomposition.
Symmetric function	A bivariate function on vectors is symmetric if $f(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}, \mathbf{u})$. For example, the inner product on real vectors is symmetric, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Symmetric matrix	A real square $n \times n$ matrix is symmetric if $a_{ij} = a_{ji}$ for all column and row indices or $A = A^T$ where A^T is the transpose of A . A real symmetric $n \times n$ matrix has n real eigenvalues when counting multiplicity.
System of linear equations	A collection of m linear equations where a solution must satisfy all m linear equations. All the linear equations must be linear equations in the same n unknowns.
Target vector	When we are given a vector $\mathbf{v} \in \mathbf{F}^m$ and a matrix $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, if we are trying to find all vectors $\mathbf{u} \in \mathbf{F}^n$ such that $A\mathbf{u} = \mathbf{v}$, we call $\mathbf{v}$ the target vector.
3-space	Another term for $\mathbf{R}^3$.
Trace	The trace of a square matrix $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ is the sum of the diagonal entries of the matrix and is represented as $\text{tr}(A)$. It happens that $\det(I + \varepsilon A) \approx 1 + \varepsilon \text{tr}(A)$
Transpose	The transpose of a $m \times n$ real matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is an $n \times m$ matrix $A^T: \mathbf{R}^m \rightarrow \mathbf{R}^n$ where the $(ij)^{\text{th}}$ entry of the transpose A^T equals a_{ji} . The transpose is that matrix such that $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^T \mathbf{v} \rangle$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Every matrix has a transpose.
Triangle inequality	The property that the distance from a to c is always less than or equal to the distance from a to b plus the distance from b to c . This is always true for the absolute value of a field $ \alpha + \beta \leq \alpha + \beta $, and it is also true for any vector norm $\ \mathbf{u} + \mathbf{v}\ \leq \ \mathbf{u}\ + \ \mathbf{v}\ $. This is also true for the distance between vectors: $\ \mathbf{u} - \mathbf{v}\ \leq \ \mathbf{u} - \mathbf{w}\ + \ \mathbf{w} - \mathbf{v}\ $, but this can be found by substituting $\mathbf{u} \leftarrow \mathbf{u} - \mathbf{w}$ and $\mathbf{v} \leftarrow \mathbf{w} - \mathbf{v}$ into the triangle inequality for vector norms.
Triangular matrix	A matrix that is either upper- or lower-triangular. If A is triangular, then the solution to $A\mathbf{u} = \mathbf{v}$ for a target vector $\mathbf{v}$ and an unknown vector $\mathbf{u}$ can be found using either backward or forward substitution, respectively.
Tri-diagonal matrix	A matrix that has all zero entries except for possibly diagonal, super-diagonal and sub-diagonal entries. The zero matrix and diagonal matrices are tri-diagonal.
Trivial solution	The trivial solution to the system of homogeneous linear equations $A\mathbf{u} = \mathbf{0}$ is the solution $\mathbf{u} = \mathbf{0}$, which is always a solution. The interesting (and therefore non-trivial) question is if there are other solutions.
Trivial subspaces	Given any vector space $\mathcal{U}$, the trivial subspaces of $\mathcal{U}$ are all of $\mathcal{U}$ and $\{\mathbf{0}_{\mathcal{U}}\}$.
2-norm	The norm induced by an inner product. For finite n -dimensional vectors, the two-norm of a real vector is $\ \mathbf{u}\ _2 = \sqrt{\sum_{k=1}^n u_k^2}$ and the two-norm of a complex vector is $\ \mathbf{u}\ _2 = \sqrt{\sum_{k=1}^n u_k ^2} = \sqrt{\sum_{k=1}^n u_k^* u_k}$.
Underdetermined system of linear equations	Any system of linear equations where the number of equations is less than the number of unknowns. In this case, there are either no solutions or infinitely many solutions. There can never be exactly one unique solution.
Unit vector	A vector with a norm equal to 1. Generally, in this course, we only use the 2-norm. Note that $\ \mathbf{u}\ _2 = 1$ if and only if $\langle \mathbf{u}, \mathbf{u} \rangle = 1$.
Unitary matrix	A square complex matrix $A: \mathbf{C}^n \rightarrow \mathbf{C}^n$ is unitary if its columns form an orthonormal basis for $\mathbf{C}^n$. Unitary matrices are the only complex matrices that are isometric, meaning that $\ A\mathbf{u}\ = \ \mathbf{u}\ $ for all vectors $\mathbf{u} \in \mathbf{C}^n$.
Unknown	Another name for a variable.
Upper-triangular matrix	A matrix is said to be upper triangular if each entry a_{ij} where $i > j$ is zero. That is, each entry that is to the left or below a diagonal entry must be zero. The zero matrix and diagonal matrices are upper-triangular matrices.
Variable	A symbol representing a real or complex number that is not necessarily fixed. The value of a variable may be constrained by a linear equation or system of linear equations. Also known as an unknown.
Vector addition	Given a collection of vectors, vector addition is an operation $\mathbf{u} + \mathbf{v}$ where this operation is commutative and associative, and where the operation of scalar multiplication distributes over this operation: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
Vector space	A vector space $\mathcal{U}$ is a collection of vectors associated with a field (either $\mathbf{R}$ or $\mathbf{C}$) where there are two operations defined: scalar multiplication and vector addition, where the axioms of a vector space are satisfied by the defined operations.

Vector space of polynomials	Given a variable x , the vector space of all polynomials $\mathbf{F}[x]$ is the collection of all polynomials with finite degree that have coefficients in $\mathbf{F}$. If you add two polynomials in $\mathbf{F}[x]$, you get a polynomial in $\mathbf{F}[x]$. If you multiply a polynomial in $\mathbf{F}[x]$ by a scalar in $\mathbf{F}$, the result is still a polynomial in $\mathbf{F}[x]$. There is a zero polynomial $z(x) = 0$. Given a polynomial p , if you multiply it by -1 , you get a polynomial $-p$ such that $p + (-p) = z$ where z is the zero polynomial.
Volume in $\mathbf{R}^n$	In $\mathbf{R}^n$, the volume of the unit box (all vectors $\mathbf{u}$ such that each entry satisfies $0 \leq u_k \leq 1$) is said to be 1. The volume of all box where each entry satisfies $0 \leq u_k \leq 0.5$ would be of volume 0.5^n . The volume of any region that is a non-trivial subspace of $\mathbf{R}^n$ is automatically zero.
Zero map	The zero map $O: \mathcal{U} \rightarrow \mathcal{V}$ is that map such that $O\mathbf{u} = \mathbf{0}_{\mathcal{V}}$ for all vectors $\mathbf{u} \in \mathcal{U}$. The null space is $\mathcal{U}$ and the range is $\{\mathbf{0}_{\mathcal{V}}\}$.
Zero matrix	$O_{m,n}$ is an $m \times n$ matrix $O_{m,n}: \mathbf{F}^n \rightarrow \mathbf{F}^m$ where each entry is 0.
Zero operator	The zero operator $O: \mathcal{U} \rightarrow \mathcal{U}$ is that map such that $O\mathbf{u} = \mathbf{0}$ for all vectors $\mathbf{u} \in \mathcal{U}$.
Zero row	A row in a matrix is described as a zero row if all the entries of that row are equal to zero.
Zero vector	In a vector space $\mathcal{U}$, the zero vector is that vector such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all vectors $\mathbf{u} \in \mathcal{U}$. For finite n -dimensional vector spaces, the zero vector $\mathbf{0}$ is that vector with all entries equal to 0. For functions, the zero vector is that function z such that $z(x) = 0$ for all x in the domain of z .

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