

NE 112 Linear algebra for nanotechnology engineering
Sections 001
PRACTICE MIDTERM EXAMINATION

- 1 [1]** Prove that the additive inverse of a point α in a field is $(-1)\alpha$.
- 2 [1]** Prove that the additive inverse of a vector $\mathbf{v}$ in a vector space is $(-1)\mathbf{v}$.
- 3 [3]** Find the polar representations of $3 - 3j$, $-4 + 4j$ and $\frac{1}{2} + \frac{\sqrt{3}}{2}j$. Please recall that $r\angle\theta = r\cos(\theta) + j\sin(\theta)$.
- 4 [1]** Find the rectangular representations of $\frac{3+4j}{4-3j}$.
- 5 [1]** Simplify the expression $\frac{-a^2c + 2abc - b^2c}{(a-b)^2}$ when $a = -523 + 215j$, $a = 559 - 126j$ and $c = -135 + 923j$ and write the answer in the rectangular representation.
- 6 [2]** If a polynomial p with real coefficients of degree 3 has the values $p(-2) = -5$, $p(0) = 4$, $p(7) = -52$, describe the possibilities for p having complex roots.
- 7 [2]** You are told that $\langle x, y \rangle = \sum_{k=0}^{\infty} x_k y_k$ where $x = (2, 1, 0.5, 0.25, 0.125, \dots)$ and $y = (-3, 1.5, 0.75, 0.375, \dots)$. Find the inner product.
- 8 [4]** Write the vector $\begin{pmatrix} \alpha + 3\beta \\ 2\beta - \alpha \\ -\beta \end{pmatrix}$ where α and β are any real numbers as a linear combination of two vectors. What are two distinct subspaces of the subspace defined by this set of vectors?
- 9 [2]** Consider all vectors of the form $\begin{pmatrix} \alpha \\ \beta \\ \alpha\beta \end{pmatrix}$ where α and β are any real numbers. Is this a subspace of $\mathbf{R}^3$?
- 10 [2]** Find two non-zero vectors that are orthogonal to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ but where the two vectors are not parallel to each other.
- 11 [2]** Prove that if $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, then $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = 0$. You may assume that these are real vectors.
- 12 [2]** Prove that the collection of all vectors that are orthogonal to a given vector $\mathbf{v}$ forms a subspace of the vector space in which $\mathbf{v}$ exists.
- 13 [3]** Describe all subspaces of $\mathbf{R}^2$. You may use images.
- 14 [1]** If $\|\mathbf{u}\|_2 = 5$ and $\|\mathbf{v}\|_2 = 6$ and you find that $\langle 2\mathbf{u}, 3\mathbf{v} \rangle = 180$, what can you say about $\mathbf{u}$ and $\mathbf{v}$?

- 14 [1]** You are told that $\left\| \begin{pmatrix} 2j \\ -2 \\ -2 \end{pmatrix} \right\|_2 = 2\sqrt{3}$ and that $\left\| \begin{pmatrix} 3j \\ 5 \\ -2 \end{pmatrix} \right\|_2 = \sqrt{38}$. Find $\left\| \begin{pmatrix} 5j \\ 3 \\ -4 \end{pmatrix} \right\|_2$ using this information.

Properties of fields:

1. A field is *closed* under addition and multiplication.
2. Addition and multiplication are *associative*: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
3. Addition and multiplication are *commutative*: $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
4. There is a 0 such that $\alpha + 0 = \alpha$.
5. There is a 1 such that $\alpha \cdot 1 = \alpha$.
6. Existence of additive inverses: given α , there is a $-\alpha$ such that $\alpha + (-\alpha) = 0$.
7. Existence of multiplicative inverses for non-zero values: given α , there is an α^{-1} such that $\alpha \cdot (\alpha^{-1}) = 1$.
8. Multiplication distributes across addition: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Properties of vector spaces: Given any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and scalars α and β , the vector space is closed under vector addition and scalar multiplication. Additionally,

1. $1 \cdot \mathbf{u} = \mathbf{u}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
6. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
7. There exists a $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
8. Given $\mathbf{u}$, there is a $-\mathbf{u}$ such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of the inner product for real vectors:

1. *symmetric* as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$,
2. *bilinear* as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the inner product for complex vectors:

1. *conjugate symmetric*, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and
2. *sesquilinear* as $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$ but $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^*\langle \mathbf{u}, \mathbf{w} \rangle + \beta^*\langle \mathbf{v}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of a norm:

1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$, and
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$$\|\mathbf{u}\|_2 \stackrel{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$

$$\sum_{k=0}^n \alpha r^k = \alpha \frac{1-r^{n+1}}{1-r} \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha r^k = \frac{\alpha}{1-r}$$