

NE 112 *Linear algebra for nanotechnology engineers*
Sections 001
PRACTICE MIDTERM EXAMINATION

- 1 [1] What is the name of the property that describes that $x(y + z) = xy + xz$?
- 2 [2] Find the polar representations of the complex sum $(3 - j) - (2 + 2j)$ and the rectangular representation of the expression $1 + j + \frac{2+j}{2-j}$.

3 [2] Evaluate $\frac{z+1}{1+\frac{1}{z}}$ when $z = -1 + j$.

- 4 [4] Simplify the expression so that the only conjugates are z^* and w^* :

$$\left(\frac{w + \frac{j}{1+w+jz}}{\frac{wz}{w+j}} \right)^*$$

You don't have to make any other simplifications.

- 5 [2] If a polynomial p with real coefficients of degree 7 has the values $p(-2) = -5$, $p(0) = 4$, $p(1) = 0$ and $p(7) = 52$, describe the possibilities for p having complex roots.

- 6 [2] You are told that $\sum_{k=0}^{\infty} r^k = 2 + j$. What is r ?

- 7 [4] Demonstrate that all vectors of the form $\begin{pmatrix} 2\alpha - \beta \\ 3\beta + \alpha \\ -\alpha \end{pmatrix}$ where α and β are any real

numbers is a subspace of $\mathbf{R}^3$. Describe two distinct subspaces of this subspace.

- 8 [2] Given the vector space of all polynomials, is the subset of all polynomials p such that $p(1) = p(-1)$ a subspace of the vector space of all polynomials?

- 9 [4] Find a vector such that $\|\mathbf{u}\|_1 = 5$ and $\|\mathbf{u}\|_{\infty} = 2$. What is the 2-norm of your vector?

- 10 [2] Simplify the expression $\left\langle \frac{1}{7+5j} \mathbf{u}, (7-5j) \mathbf{v} \right\rangle$ as much as possible.

- 11 [4] Is it ever possible that two infinite sequences $(1, a, a^2, a^3, a^4, \dots)$ and $(1, b, b^2, b^3, b^4, \dots)$ to be orthogonal? Justify your answer.

- 12 [2] Prove that $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}(\mathbf{v})) = \text{proj}_{\mathbf{u}}(\mathbf{v})$; that is, if you project a second time onto the same vector, the result is unchanged.

- 13 [2] Given two complex unit vectors $\mathbf{u}$ and $\mathbf{v}$ (vectors with 2-norm equal to 1), what is the maximum possible value of $|\langle 2\mathbf{u}, 3j\mathbf{v} \rangle|$?

- 14 [3] What row operation undoes the operation $R_{3,2;5 \rightarrow 2}$? What row operation undoes the operation $R_{3 \leftrightarrow 1}$?

- 15 [2] In your own words, define when two augmented matrices are row equivalent.

- 16 [3] Find the rank of the matrix $\begin{pmatrix} 5 & 3 & 2 & 1 \\ 2 & 1.2 & 3.8 & 5.4 \\ -4.5 & -2.7 & 0 & 4.1 \end{pmatrix}$.

Properties of fields:

1. A field is *closed* under addition and multiplication.
2. Addition and multiplication are *associative*: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
3. Addition and multiplication are *commutative*: $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
4. There is a 0 such that $\alpha + 0 = \alpha$.
5. There is a 1 such that $\alpha \cdot 1 = \alpha$.
6. Existence of additive inverses: given α , there is a $-\alpha$ such that $\alpha + (-\alpha) = 0$.
7. Existence of multiplicative inverses for non-zero values: given α , there is an α^{-1} such that $\alpha \cdot (\alpha^{-1}) = 1$.
8. Multiplication distributes across addition: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Properties of vector spaces: Given any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and scalars α and β , the vector space is closed under vector addition and scalar multiplication. Additionally,

1. $1 \cdot \mathbf{u} = \mathbf{u}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
6. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
7. There exists a $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
8. Given $\mathbf{u}$, there is a $-\mathbf{u}$ such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of the inner product for real vectors:

1. *symmetric* as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$,
2. *bilinear* as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the inner product for complex vectors:

1. *conjugate symmetric*, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and
2. *sesquilinear* as $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$ but $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^*\langle \mathbf{u}, \mathbf{w} \rangle + \beta^*\langle \mathbf{v}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of a norm:

1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$, and
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$$\|\mathbf{u}\|_2 \stackrel{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}$$