

An $m \times n$ matrix A is a mapping $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ and is represented by

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}.$$

For example, we will consider the following six matrices

$$\begin{aligned} B: \mathbf{R}^3 \rightarrow \mathbf{R}^4 & \quad D: \mathbf{R}^3 \rightarrow \mathbf{R}^3 & \quad F: \mathbf{R}^5 \rightarrow \mathbf{R}^3 \\ B = \begin{pmatrix} -0.4 & -2.6 & -2.1 \\ -1.4 & -2.9 & -3.7 \\ 2.0 & 3.0 & 1.0 \\ 0.2 & 4.3 & 5.1 \end{pmatrix} & \quad D = \begin{pmatrix} 1.2 & -1.2 & -2.1 \\ 4.0 & 3.0 & -2.0 \\ -2.0 & 1.5 & 6.0 \end{pmatrix} & \quad F = \begin{pmatrix} -0.8 & 2.8 & -1.4 & 1.0 & 0.4 \\ 1.6 & -2.6 & 1.3 & 4.5 & 3.2 \\ 2.0 & -2.0 & 1.0 & 0.0 & 4.0 \end{pmatrix} \\ C: \mathbf{R}^4 \rightarrow \mathbf{R}^5 & \quad E: \mathbf{R}^4 \rightarrow \mathbf{R}^4 & \quad G: \mathbf{R}^6 \rightarrow \mathbf{R}^4 \\ C = \begin{pmatrix} -1.0 & -0.4 & -2.0 & -2.6 \\ 5.0 & 2.0 & 4.0 & -2.0 \\ 3.0 & 1.2 & 0.8 & -5.2 \\ -2.5 & -1.0 & 0.0 & 6.0 \\ 2.0 & 0.8 & 2.0 & 0.2 \end{pmatrix} & \quad E = \begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 \\ 5.0 & 3.0 & -2.0 & 3.0 \\ -1.5 & 3.1 & -1.4 & 0.1 \\ -0.5 & 2.5 & 0.8 & 3.4 \end{pmatrix} & \quad G = \begin{pmatrix} -3.5 & -1.4 & 4.1 & 0.6 & 0.3 & -3.4 \\ 2.0 & 0.8 & -0.4 & 1.6 & -0.7 & -0.3 \\ -1.5 & -0.6 & 0.7 & -0.8 & 4.6 & 0.6 \\ 5.0 & 2.0 & -3.0 & 2.0 & 1.0 & 2.0 \end{pmatrix} \end{aligned}$$

If $m = n$, the matrix is said to be a *linear operator*, otherwise the matrix is said to be a *linear map*.

Defining matrix-vector multiplication

The matrix-vector product $A\mathbf{u}$ is defined as the linear combination of the columns of A with the coefficients taken from the vector $\mathbf{u}$; that is, if

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$$

then

$$A\mathbf{u} = u_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + u_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ \vdots \\ a_{m,2} \end{pmatrix} + u_3 \begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \\ \vdots \\ a_{m,3} \end{pmatrix} + \cdots + u_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \\ \vdots \\ a_{m,n} \end{pmatrix}.$$

For example, given our example matrices,

$$\text{If } \mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \in \mathbf{R}^3, \quad B\mathbf{u} = 2 \begin{pmatrix} -0.4 \\ -1.4 \\ 2.0 \\ 0.2 \end{pmatrix} + 3 \begin{pmatrix} -2.6 \\ -2.9 \\ 3.0 \\ 4.3 \end{pmatrix} + (-1) \begin{pmatrix} -2.1 \\ -3.7 \\ 1.0 \\ 5.1 \end{pmatrix} = \begin{pmatrix} -6.5 \\ -7.8 \\ 12.0 \\ 8.2 \end{pmatrix} \in \mathbf{R}^4.$$

$$\text{If } \mathbf{u} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 2 \end{pmatrix} \in \mathbf{R}^4, \quad C\mathbf{u} = 3 \begin{pmatrix} -1.0 \\ 5.0 \\ 3.0 \\ -2.5 \\ 2.0 \end{pmatrix} + (-2) \begin{pmatrix} -0.4 \\ 2.0 \\ 1.2 \\ -1.0 \\ 0.8 \end{pmatrix} + 1 \begin{pmatrix} -2.0 \\ 4.0 \\ 0.8 \\ 0.0 \\ 2.0 \end{pmatrix} + 2 \begin{pmatrix} -2.6 \\ -2.0 \\ -5.2 \\ 6.0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} -9.4 \\ 11.0 \\ -3.0 \\ 6.5 \\ 6.8 \end{pmatrix} \in \mathbf{R}^5.$$

$$\text{If } \mathbf{u} = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} \in \mathbf{R}^3, \quad D\mathbf{u} = (-3) \begin{pmatrix} 1.2 \\ 4.0 \\ -2.0 \end{pmatrix} + 2 \begin{pmatrix} 1.2 \\ 3.0 \\ 1.5 \end{pmatrix} + 4 \begin{pmatrix} -2.1 \\ -2.0 \\ 6.0 \end{pmatrix} = \begin{pmatrix} -14.4 \\ -14.0 \\ 33.0 \end{pmatrix} \in \mathbf{R}^3.$$

$$\text{If } \mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -3 \end{pmatrix} \in \mathbf{R}^4, \quad E\mathbf{u} = 2 \begin{pmatrix} 1.0 \\ 5.0 \\ -1.5 \\ -0.5 \end{pmatrix} + 1 \begin{pmatrix} -0.2 \\ -3.0 \\ 3.1 \\ 2.5 \end{pmatrix} + 1 \begin{pmatrix} 1.0 \\ -2.0 \\ -1.4 \\ 0.8 \end{pmatrix} + (-3) \begin{pmatrix} 1.9 \\ 3.0 \\ 0.1 \\ 3.4 \end{pmatrix} = \begin{pmatrix} -2.9 \\ 2.0 \\ -1.6 \\ -7.9 \end{pmatrix} \in \mathbf{R}^4.$$

$$\text{If } \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \\ -3 \end{pmatrix} \in \mathbf{R}^5, \quad F\mathbf{u} = 2 \begin{pmatrix} -0.8 \\ 1.6 \\ 2.0 \end{pmatrix} + (-1) \begin{pmatrix} 2.8 \\ -2.6 \\ -2.0 \end{pmatrix} + 2 \begin{pmatrix} -1.4 \\ 1.3 \\ 1.0 \end{pmatrix} + 1 \begin{pmatrix} 1.0 \\ 4.5 \\ 0.0 \end{pmatrix} + (-3) \begin{pmatrix} 0.4 \\ 3.2 \\ 4.0 \end{pmatrix} = \begin{pmatrix} -7.4 \\ 3.3 \\ -4.0 \end{pmatrix} \in \mathbf{R}^3.$$

$$\text{If } \mathbf{u} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \in \mathbf{R}^6, \quad G\mathbf{u} = (-2) \begin{pmatrix} -3.5 \\ 2.0 \\ -1.5 \\ 5.0 \end{pmatrix} + (-1) \begin{pmatrix} -1.4 \\ 0.8 \\ -0.6 \\ 2.0 \end{pmatrix} + 1 \begin{pmatrix} 4.1 \\ -0.4 \\ 0.7 \\ -3.0 \end{pmatrix} + 0 \begin{pmatrix} 0.6 \\ 1.6 \\ -0.8 \\ 2.0 \end{pmatrix} + 2 \begin{pmatrix} 0.3 \\ -0.7 \\ 4.6 \\ 1.0 \end{pmatrix} + 1 \begin{pmatrix} -3.4 \\ -0.3 \\ 0.6 \\ 2.0 \end{pmatrix} = \begin{pmatrix} -14.5 \\ 0.2 \\ 9.9 \\ 15.0 \end{pmatrix} \in \mathbf{R}^4.$$

Terminology: image and pre-image

The *image* of a vector $\mathbf{u} \in \mathbf{R}^n$ under the mapping A is that vector $A\mathbf{u} \in \mathbf{R}^m$.

Thus,

1. the image of $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ under B is $\begin{pmatrix} -6.5 \\ -7.8 \\ 12.0 \\ 8.2 \end{pmatrix}$,

2. the image of $\begin{pmatrix} 3 \\ -2 \\ 1 \\ 2 \end{pmatrix}$ under C is $\begin{pmatrix} -9.4 \\ 11.0 \\ -3.0 \\ 6.5 \\ 6.8 \end{pmatrix}$,

3. the image of $\begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$ under D is $\begin{pmatrix} -14.4 \\ -14.0 \\ 33.0 \end{pmatrix}$,

4. the image of $\begin{pmatrix} 2 \\ 1 \\ 1 \\ -3 \end{pmatrix}$ under E is $\begin{pmatrix} -2.9 \\ 2.0 \\ -1.6 \\ -7.9 \end{pmatrix}$,

5. the image of $\begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \\ -3 \end{pmatrix}$ under F is $\begin{pmatrix} -7.4 \\ 3.3 \\ -4.0 \end{pmatrix}$, and

6. the image of $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}$ under G is $\begin{pmatrix} -14.5 \\ 0.2 \\ 9.9 \\ 15.0 \end{pmatrix}$.

If $\mathbf{v} \in \mathbf{R}^m$ and there is a vector $\mathbf{u} \in \mathbf{R}^n$ such that $A\mathbf{u} = \mathbf{v}$, we say that $\mathbf{u}$ is a *pre-image* of $\mathbf{v}$.

Finding pre-images of a vector

The pre-image of $\mathbf{v}$ is the set of all vectors in $\mathbf{u} \in \mathbf{R}^n$ such that $A\mathbf{u} = \mathbf{v}$. We can find the pre-image of a vector $\mathbf{v}$ by finding all solutions to

$$(A \mid \mathbf{v}) = \left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & v_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & v_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} & v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} & v_m \end{array} \right).$$

For our six examples, the pre-images of the vectors we have found will, in some cases, be just that vector, and in others, will be a set of vectors containing the vector in question.

Example 1: $B: \mathbf{R}^3 \rightarrow \mathbf{R}^4$

For the map B , given $\mathbf{v} = \begin{pmatrix} -6.5 \\ -7.8 \\ 12.0 \\ 8.2 \end{pmatrix}$, we have $\begin{pmatrix} -0.4 & -2.6 & -2.1 & -6.5 \\ -1.4 & -2.9 & -3.7 & -7.8 \\ 2.0 & 3.0 & 1.0 & 12.0 \\ 0.2 & 4.3 & 5.1 & 8.2 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 & 12 \\ 0 & 4 & 5 & 7 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and thus we note

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, and
2. the rank of the matrix equals the number of columns, so a unique solution exists.

Therefore, we can find this unique pre-image of $\mathbf{v}$ using backward substitution; that is, $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

If, however, we attempt to find a pre-image for $\mathbf{v} = \begin{pmatrix} -5.5 \\ -7.8 \\ 12.0 \\ 8.2 \end{pmatrix}$, we see that

$\begin{pmatrix} -0.4 & -2.6 & -2.1 & -5.5 \\ -1.4 & -2.9 & -3.7 & -7.8 \\ 2.0 & 3.0 & 1.0 & 12.0 \\ 0.2 & 4.3 & 5.1 & 8.2 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 & 12 \\ 0 & 4 & 5 & 7 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and thus no solution exists as the rank of the matrix is 3, while

the rank of the augmented matrix is 4, and thus this vector $\mathbf{v}$ has no pre-image.

Example 2: $C: \mathbf{R}^4 \rightarrow \mathbf{R}^5$

For the map C , given $\mathbf{v} = \begin{pmatrix} -9.4 \\ 11.0 \\ -3.0 \\ 6.5 \\ 6.8 \end{pmatrix}$, we have $\begin{pmatrix} -1.0 & -0.4 & -2.0 & -2.6 & -9.4 \\ 5.0 & 2.0 & 4.0 & -2.0 & 11.0 \\ 3.0 & 1.2 & 0.8 & -5.2 & -3.0 \\ -2.5 & -1.0 & 0.0 & 6.0 & 6.5 \\ 2.0 & 0.8 & 2.0 & 0.2 & 6.8 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 4 & -2 & 11 \\ 0 & 0 & 2 & 5 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Again,

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, but
2. the rank of the matrix is two less than the number of columns, so there are two free variables.

Thus, assigning free variables to Columns 2 and 4, as these columns do not have leading non-zero entries, we have that the solutions are $\alpha_3 = 6 - 2.5\alpha_4, \alpha_1 = -2.6 - 0.4\alpha_2 + 2.4\alpha_4$, so the collection of all solutions is of the form

$$\begin{pmatrix} -2.6 \\ 0 \\ 6.0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 2.4 \\ 0 \\ -2.5 \\ 1 \end{pmatrix}.$$

This does not form a subspace in $\mathbf{R}^4$. You will note, however, that

$$\begin{pmatrix} -2.6 \\ 0 \\ 6.0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 2.4 \\ 0 \\ -2.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 2 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} -12.2 \\ 10.0 \\ -7.6 \\ 12.0 \\ 7.4 \end{pmatrix}$ to be the image, we see that it is in the set of all

solutions.

If, however, we attempt to find the pre-image for $\mathbf{v} = \begin{pmatrix} -12.2 \\ 10.0 \\ -6.6 \\ 12.0 \\ 7.4 \end{pmatrix}$, we have

$$\begin{pmatrix} -1.0 & -0.4 & -2.0 & -2.6 & -12.4 \\ 5.0 & 2.0 & 4.0 & -2.0 & 11.0 \\ 3.0 & 1.2 & 0.8 & -5.2 & -3.0 \\ -2.5 & -1.0 & 0.0 & 6.0 & 6.5 \\ 2.0 & 0.8 & 2.0 & 0.2 & 6.8 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 4 & -2 & 11 \\ 0 & 0 & 2 & 5 & 12 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and thus the rank of the matrix is one less than rank of the augmented matrix, so no solution exists.

Example 3: $D:\mathbf{R}^3 \rightarrow \mathbf{R}^3$

For the operator D , given $\mathbf{v} = \begin{pmatrix} -14.4 \\ -14.0 \\ 33.0 \end{pmatrix}$, we have $\left(\begin{array}{ccc|c} 1.2 & -1.2 & -2.1 & -14.4 \\ 4.0 & 3.0 & -2.0 & -14.0 \\ -2.0 & 1.5 & 6.0 & 33.0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 4 & 3 & -2 & -14 \\ 0 & 3 & 5 & 26 \\ 0 & 0 & 2 & 8 \end{array} \right)$ and thus, as

the rank of the augmented matrix equals the number of columns, there is a unique solution, and we can find that unique pre-image using backward substitution to find $\mathbf{u} = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$. Because the rank of the matrix equals the number

of rows which equals the dimension of the co-domain, there is a unique pre-image for every vector in $\mathbf{R}^3$.

Example 4: $E: \mathbf{R}^4 \rightarrow \mathbf{R}^4$

For the operator E , given $\mathbf{v} = \begin{pmatrix} -2.9 \\ 2.0 \\ -1.6 \\ -7.9 \end{pmatrix}$, we have $\begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 & -2.9 \\ 5.0 & 3.0 & -2.0 & 3.0 & 2.0 \\ -1.5 & 3.1 & -1.4 & 0.1 & -1.6 \\ -0.5 & 2.5 & 0.8 & 3.4 & -7.9 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -2 & 3 & 2 \\ 0 & 4 & -2 & 1 & -1 \\ 0 & 0 & 2 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

From the row-echelon form, we have

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, but
2. the rank of the matrix is one less than the number of columns, so there is one free variable.

Thus, assigning a free variables to Column 4, as this column does not have a leading non-zero entry, we have that the solutions are $\alpha_3 = -3.5 - 1.5\alpha_4, \alpha_2 = -2 - \alpha_4, \alpha_1 = 0.2 - 0.6\alpha_4$, so the collection of all solutions is of the form

$$\begin{pmatrix} 0.2 \\ -2.0 \\ -3.5 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} -0.6 \\ -1.0 \\ -1.5 \\ 1 \end{pmatrix}.$$

This does not form a subspace in $\mathbf{R}^4$. You will note, however, that

$$\begin{pmatrix} 0.2 \\ -2.0 \\ -3.5 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} -0.6 \\ -1.0 \\ -1.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -3 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} -2.9 \\ 2.0 \\ -1.6 \\ -7.9 \end{pmatrix}$ to be the image, we see that it is in the set of all solutions.

If, however, we attempt to find the pre-image for $\mathbf{v} = \begin{pmatrix} -2.9 \\ 2.0 \\ -1.6 \\ -7.9 \end{pmatrix}$, we have

$$\begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 & -0.9 \\ 5.0 & 3.0 & -2.0 & 3.0 & 2.0 \\ -1.5 & 3.1 & -1.4 & 0.1 & -1.6 \\ -0.5 & 2.5 & 0.8 & 3.4 & -7.9 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -2 & 3 & 2 \\ 0 & 4 & -2 & 1 & -1 \\ 0 & 0 & 2 & 3 & -7 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

and thus the rank of the matrix is one less than rank of the augmented matrix, so no solution exists.

Example 5: $F: \mathbb{R}^5 \rightarrow \mathbb{R}^3$

For the map F , given $\mathbf{v} = \begin{pmatrix} -7.4 \\ 3.3 \\ -4.0 \end{pmatrix}$, we have,

$$\left(\begin{array}{ccccc|c} -0.8 & 2.8 & -1.4 & 1.0 & 0.4 & -7.4 \\ 1.6 & -2.6 & 1.3 & 4.5 & 3.2 & 3.3 \\ 2.0 & -2.0 & 1.0 & 0.0 & 4.0 & -4.0 \end{array} \right) \sim \left(\begin{array}{ccccc|c} 2 & -2 & 1 & 0 & 4 & -4 \\ 0 & 2 & -1 & 1 & 2 & -9 \\ 0 & 0 & 0 & 5 & 1 & 2 \end{array} \right).$$

From the row-echelon form, we have,

1. the rank of the matrix equals the dimension of the co-domain, so it is always guaranteed that at least one solution exists, and
2. the rank of the matrix is two less than the number of columns, so there are two free variables.

Thus, assigning free variables to Columns 3 and 5, as these columns do not have leading non-zero entries, we have that the solutions are $\alpha_4 = 0.4 - 0.2\alpha_5$, $\alpha_2 = -4.7 + 0.5\alpha_3 - 0.9\alpha_5$, $\alpha_1 = -6.7 - 2.9\alpha_5$, so the collection of all solutions is of the form

$$\begin{pmatrix} -6.7 \\ -4.7 \\ 0 \\ 0.4 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} -2.9 \\ -0.9 \\ 0 \\ -0.2 \\ 1 \end{pmatrix}.$$

This does not form a subspace in $\mathbb{R}^4$. You will note, however, that

$$\begin{pmatrix} -6.7 \\ -4.7 \\ 0 \\ 0.4 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} -2.9 \\ -0.9 \\ 0 \\ -0.2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \\ -3 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} -7.4 \\ 3.3 \\ -4.0 \end{pmatrix}$ to be the image, we see that it is in the set of all solutions.

Because the rank of the matrix equals the number of rows (which equals dimension of the co-domain), there will always exist infinitely many solutions for every target vector.

Example 6: $G: \mathbf{R}^6 \rightarrow \mathbf{R}^4$

For the map G , given $\mathbf{v} = \begin{pmatrix} -14.5 \\ 0.2 \\ 9.9 \\ 15.0 \end{pmatrix}$, we have,

$$\left(\begin{array}{cccccc|c} -3.5 & -1.4 & 4.1 & 0.6 & 0.3 & -3.4 & -14.5 \\ 2.0 & 0.8 & -0.4 & 1.6 & -0.7 & -0.3 & 0.2 \\ -1.5 & -0.6 & 0.7 & -0.8 & 4.6 & 0.6 & 9.9 \\ 5.0 & 2.0 & -3.0 & 2.0 & 1.0 & 2.0 & 15.0 \end{array} \right) \sim \left(\begin{array}{cccccc|c} 5 & 2 & -3 & 2 & 1 & 2 & 15 \\ 0 & 0 & 2 & 2 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 5 & 1 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

From the row-echelon form, we have,

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, but
2. the rank of the matrix is three less than the number of columns, so there are three free variables.

Thus, assigning free variables to Columns 2, 4 and 6, as these columns do not have leading non-zero entries, we have that the solutions are $\alpha_5 = 2.8 - 0.2\alpha_6$, $\alpha_3 = -3.4 - \alpha_4 + 1.1\alpha_6$, $\alpha_1 = 0.4 - 0.4\alpha_2 - \alpha_4 + 0.3\alpha_6$, so the collection of all solutions is of the form

$$\begin{pmatrix} 0.4 \\ 0 \\ -3.4 \\ 0 \\ 2.8 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} -1.0 \\ 0 \\ -1.0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_6 \begin{pmatrix} 0.3 \\ 0 \\ 1.1 \\ 0 \\ -0.2 \\ 1 \end{pmatrix}$$

This does not form a subspace in $\mathbf{R}^6$. You will note, however, that

$$\begin{pmatrix} 0.4 \\ 0 \\ -3.4 \\ 0 \\ 2.8 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -1.0 \\ 0 \\ -1.0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0.3 \\ 0 \\ 1.1 \\ 0 \\ -0.2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 2 \\ 4 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} -14.5 \\ 0.2 \\ 9.9 \\ 15.0 \end{pmatrix}$ to be the image, we see that it is in the set of all

solutions.

A basis for the range

We have seen that all linear combinations of vectors forms a sub-space of a vector space. Matrix-vector multiplication is taking a linear combination of the columns of the matrix, and if you consider the image of every vector in the domain, you are essentially finding all linear combinations of the columns in the co-domain. We call the image of all the vectors in the domain the range, and the range is a subspace of the co-domain.

Thus, if the columns of the matrix are linearly independent, they form a basis for the image of the domain. If the columns are not linearly independent, finding the row-equivalent row-echelon form of the matrix allows you to choose those columns that are linearly independent, and these linearly independent columns form a basis for the image of the domain. The range of A is all of $\mathbf{R}^m$ if and only if $\text{rank}(A) = m$, the dimension of the co-domain.

If the dimension of the domain is less than the dimension of the co-domain, the range will always be a proper subspace of the co-domain (that is, the range will never equal the entire co-domain).

Example 1: $B: \mathbf{R}^3 \rightarrow \mathbf{R}^4$

For the map B , we have
$$\begin{pmatrix} -0.4 & -2.6 & -2.1 \\ -1.4 & -2.9 & -3.7 \\ 2.0 & 3.0 & 1.0 \\ 0.2 & 4.3 & 5.1 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$
 and thus the dimension of the range is 3 (being the

rank of the matrix B), and a basis for the range is

$$\begin{pmatrix} -0.4 \\ -1.4 \\ 2.0 \\ 0.2 \end{pmatrix}, \begin{pmatrix} -2.6 \\ -2.9 \\ 3.0 \\ 4.3 \end{pmatrix}, \begin{pmatrix} -2.1 \\ -3.7 \\ 1.0 \\ 5.1 \end{pmatrix}.$$

Because the dimension of the range is less than the dimension of the co-domain, there are vectors in the co-domain that have no pre-image.

Example 2: $C: \mathbf{R}^4 \rightarrow \mathbf{R}^5$

For the map C , we have $\begin{pmatrix} -1.0 & -0.4 & -2.0 & -2.6 \\ 5.0 & 2.0 & 4.0 & -2.0 \\ 3.0 & 1.2 & 0.8 & -5.2 \\ -2.5 & -1.0 & 0.0 & 6.0 \\ 2.0 & 0.8 & 2.0 & 0.2 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 4 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and thus the dimension of the range is 2

(being the rank of the matrix C), and a basis for the range is

$$\begin{pmatrix} -1.0 \\ 5.0 \\ 3.0 \\ -2.5 \\ 2.0 \end{pmatrix}, \begin{pmatrix} -2.0 \\ 4.0 \\ 0.8 \\ 0.0 \\ 2.0 \end{pmatrix}.$$

Because the dimension of the range is less than the dimension of the co-domain, there are vectors in the co-domain that have no pre-image. In this case, the range forms a plane in $\mathbf{R}^5$.

Example 3: $D: \mathbf{R}^3 \rightarrow \mathbf{R}^3$

For the operator D , we have $\begin{pmatrix} 1.2 & -1.2 & -2.1 \\ 4.0 & 3.0 & -2.0 \\ -2.0 & 1.5 & 6.0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ and thus, and thus the dimension of the range is 3

(being the rank of the matrix D), and a basis for this range are the three vectors

$$\begin{pmatrix} 1.2 \\ 4.0 \\ -2.0 \end{pmatrix}, \begin{pmatrix} -1.3 \\ 3.0 \\ 1.5 \end{pmatrix}, \begin{pmatrix} -2.1 \\ -2.0 \\ 6.0 \end{pmatrix}$$

Because the dimension of the co-domain is equal to the dimension of the range, this means that every vector in the co-domain has a pre-image. In addition, as each pre-image is unique, this indicates that every vector in the domain has one image, and each vector in the co-domain has exactly one pre-image.

Example 4: $E: \mathbf{R}^4 \rightarrow \mathbf{R}^4$

For the operator E , we have $\begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 \\ 5.0 & 3.0 & -2.0 & 3.0 \\ -1.5 & 3.1 & -1.4 & 0.1 \\ -0.5 & 2.5 & 0.8 & 3.4 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -2 & 3 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Because the dimension of the range is less than the dimension of the co-domain, there are vectors in the co-domain that have no pre-image. In this case, the range forms a 3-dimensional subspace in $\mathbf{R}^4$. Every vector that has a pre-image, therefore, has infinitely many pre-images. A basis for the range may be found by choosing those vectors corresponding to columns containing leading non-zero entries in the row-echelon form, namely, Columns 1, 2 and 3. Thus, such a basis is

$$\begin{pmatrix} 1.0 \\ 5.0 \\ -1.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} -0.2 \\ 3.0 \\ 3.1 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 1.0 \\ -2.0 \\ -1.4 \\ 0.8 \end{pmatrix}.$$

Example 5: $F: \mathbb{R}^5 \rightarrow \mathbb{R}^3$

For the map F , we have $\begin{pmatrix} -0.8 & 2.8 & -1.4 & 1.0 & 0.4 \\ 1.6 & -2.6 & 1.3 & 4.5 & 3.2 \\ 2.0 & -2.0 & 1.0 & 0.0 & 4.0 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 1 & 0 & 4 \\ 0 & 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix}.$

From the row-echelon form, we have, that the rank is 3, so because the dimension of the co-domain is 3, this means that each vector in the co-domain has at least one pre-image. Because the rank is less than the number of column vectors, there are infinitely many solutions. From the row-echelon form, a basis for the range are Columns 1, 2 and 4; that is, those columns that have leading non-zero entries in the row-equivalent row-echelon form. Thus, the basis is

$$\begin{pmatrix} -0.8 \\ 1.6 \\ 2.0 \end{pmatrix}, \begin{pmatrix} 2.8 \\ -2.6 \\ -2.0 \end{pmatrix}, \begin{pmatrix} 1.0 \\ 4.5 \\ 0.0 \end{pmatrix}.$$

Example 6: $G: \mathbb{R}^6 \rightarrow \mathbb{R}^4$

For the map G , we have

$$\begin{pmatrix} -3.5 & -1.4 & 4.1 & 0.6 & 0.3 & -3.4 \\ 2.0 & 0.8 & -0.4 & 1.6 & -0.7 & -0.3 \\ -1.5 & -0.6 & 0.7 & -0.8 & 4.6 & 0.6 \\ 5.0 & 2.0 & -3.0 & 2.0 & 1.0 & 2.0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & -3 & 2 & 1 & 2 \\ 0 & 0 & 2 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the row-echelon form, we see that the dimension of the range is 3, and a basis for that range are the columns corresponding to those columns in the row-equivalent row-echelon form with leading non-zero entries; in this case, Columns 1, 3 and 5:

$$\begin{pmatrix} -3.5 \\ 2.0 \\ -1.5 \\ 5.0 \end{pmatrix}, \begin{pmatrix} 4.1 \\ -0.4 \\ 0.7 \\ -3.0 \end{pmatrix}, \begin{pmatrix} 0.3 \\ -0.7 \\ 4.6 \\ 1.0 \end{pmatrix}.$$

As the dimension of the co-domain is 4, and the dimension of the range is 3, this indicates that there are vectors in the co-domain that have no pre-image under this mapping. Also, because the rank of the matrix is less than the number of columns in the matrix, this indicates that there are infinitely many pre-images for each vector in the range.

The null space and a basis for the null space

Given a linear mapping or operator A , the set of all vectors that map onto the zero vector in the co-domain is described as the *null space* of the linear operator. To find the null space of a linear operator, we must find all solutions to $(A \mid \mathbf{0}_m)$. The null space is just the zero-vector of the domain if the rank of the matrix equals the number of columns; otherwise, there are infinitely many vectors in the null space. The null space always contains the zero vector, for $A\mathbf{0}_n = \mathbf{0}_m$. The null space is always a subspace of the domain.

Example 1: $B: \mathbf{R}^3 \rightarrow \mathbf{R}^4$

For the map B , we have
$$\left(\begin{array}{cccc|c} -0.4 & -2.6 & -2.1 & 0 & 0 \\ -1.4 & -2.9 & -3.7 & 0 & 0 \\ 2.0 & 3.0 & 1.0 & 0 & 0 \\ 0.2 & 4.3 & 5.1 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$
 and thus we note because the rank equals the

number of columns, a unique solution exists. Therefore, as we already know that the zero-vector in $\mathbf{R}^n$ maps onto the zero-vector of $\mathbf{R}^m$, this must be the unique vector in the null space. Note that the zero vector is a subspace of $\mathbf{R}^n$.

Example 2: $C: \mathbf{R}^4 \rightarrow \mathbf{R}^5$

For the map C , we have
$$\left(\begin{array}{cccc|c} -1.0 & -0.4 & -2.0 & -2.6 & 0 \\ 5.0 & 2.0 & 4.0 & -2.0 & 0 \\ 3.0 & 1.2 & 0.8 & -5.2 & 0 \\ -2.5 & -1.0 & 0.0 & 6.0 & 0 \\ 2.0 & 0.8 & 2.0 & 0.2 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 5 & 2 & 4 & -2 & 0 \\ 0 & 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$
 Here now, however, because the

rank of the matrix is less than the number of columns, there are an infinitely many vectors that map onto the zero vector $\mathbf{0}_m$. We can find all these solutions using backward substitution: $\alpha_3 = -2.5\alpha_4, \alpha_1 = -0.4\alpha_2 + 2.4\alpha_4$, so the collection of all solutions is of the form

$$\alpha_2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 2.4 \\ 0 \\ -2.5 \\ 1 \end{pmatrix}.$$

You may note that all linear combinations of these two vectors forms a subspace of $\mathbf{R}^4$, and because these are linearly independent, they form a basis, so that subspace is a plane in $\mathbf{R}^4$.

For example, you will note that the image of the vector

$$0.2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 0.3 \begin{pmatrix} 2.4 \\ 0 \\ -2.5 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.8 \\ 0.2 \\ 0.75 \\ -0.3 \end{pmatrix}$$

is the zero vector.

Example 3: $D:\mathbf{R}^3 \rightarrow \mathbf{R}^3$

For the operator D , we have $\left(\begin{array}{ccc|c} 1.2 & -1.2 & -2.1 & 0 \\ 4.0 & 3.0 & -2.0 & 0 \\ -2.0 & 1.5 & 6.0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 3 & -2 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$ and the rank of D equals the number of columns, so a unique solution exists, and thus the only pre-image of the zero vector is the zero vector itself.

Example 4: $E:\mathbf{R}^4 \rightarrow \mathbf{R}^4$

For the operator E , we have $\left(\begin{array}{cccc|c} 1.0 & -0.2 & 1.0 & 1.9 & 0 \\ 5.0 & 3.0 & -2.0 & 3.0 & 0 \\ -1.5 & 3.1 & -1.4 & 0.1 & 0 \\ -0.5 & 2.5 & 0.8 & 3.4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 5 & 3 & -2 & 3 & 0 \\ 0 & 4 & -2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$, and since the rank of the

matrix is 3, which is less than the number of columns, there are infinitely many solutions to this problem. We can use backward substitution to find that the solutions are $\alpha_3 = -1.5\alpha_4, \alpha_2 = -\alpha_4, \alpha_1 = -0.6\alpha_4$, so the collection of all solutions to the homogeneous system of linear equations is of the form

$$\alpha_4 \begin{pmatrix} -0.6 \\ -1.0 \\ -1.5 \\ 1 \end{pmatrix}.$$

This forms a 1-dimensional sub-space of $\mathbf{R}^4$; that is, a line passing through the origin in $\mathbf{R}^4$. You will note that, for example,

$$(-0.7) \begin{pmatrix} -0.6 \\ -1.0 \\ -1.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.7 \\ 1.05 \\ -0.7 \end{pmatrix}$$

maps onto the zero vector.

Example 5: $F:\mathbf{R}^5 \rightarrow \mathbf{R}^3$

For the map F , we have $\left(\begin{array}{ccccc|c} -0.8 & 2.8 & -1.4 & 1.0 & 0.4 & 0 \\ 1.6 & -2.6 & 1.3 & 4.5 & 3.2 & 0 \\ 2.0 & -2.0 & 1.0 & 0.0 & 4.0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 2 & -2 & 1 & 0 & 4 & 0 \\ 0 & 2 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \end{array}\right)$, we see that there are

infinitely many solutions to the homogeneous equation, as the rank is greater than the number of columns. Using backward substitution, we can find all such solutions, namely $\alpha_4 = -0.2\alpha_5, \alpha_2 = 0.5\alpha_3 - 0.9\alpha_5, \alpha_1 = -2.9\alpha_5$, so the collection of all solutions is of the form

$$\alpha_3 \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} -2.9 \\ -0.9 \\ 0 \\ -0.2 \\ 1 \end{pmatrix}.$$

This is a 2-dimensional subspace of $\mathbf{R}^5$. You will note that

$$-0.2 \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0.3 \begin{pmatrix} -2.9 \\ -0.9 \\ 0 \\ -0.2 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.87 \\ -0.37 \\ -0.2 \\ -0.06 \\ 0.3 \end{pmatrix}$$

does map to the zero vector $\mathbf{0}_3$ under this linear mapping.

Example 6: $G: \mathbf{R}^6 \rightarrow \mathbf{R}^4$

For the map G , we have,
$$\begin{pmatrix} -3.5 & -1.4 & 4.1 & 0.6 & 0.3 & -3.4 & 0 \\ 2.0 & 0.8 & -0.4 & 1.6 & -0.7 & -0.3 & 0 \\ -1.5 & -0.6 & 0.7 & -0.8 & 4.6 & 0.6 & 0 \\ 5.0 & 2.0 & -3.0 & 2.0 & 1.0 & 2.0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & -3 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
 and as the

rank of the matrix is 3, which is less than the number of columns, there are infinitely many solutions. We can use backward substitution to find all these solutions: $\alpha_5 = -0.2\alpha_6$, $\alpha_3 = -\alpha_4 + 1.1\alpha_6$, $\alpha_1 = -0.4\alpha_2 - \alpha_4 - 0.3\alpha_6$, so the collection of all solutions is of the form

$$\alpha_2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} -1.0 \\ 0 \\ -1.0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_6 \begin{pmatrix} -0.3 \\ 0 \\ 1.1 \\ 0 \\ -0.2 \\ 1 \end{pmatrix}$$

This is a 3-dimensional subspace in $\mathbf{R}^6$. You will note that, for example,

$$0.3 \begin{pmatrix} -0.4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.7 \begin{pmatrix} -1.0 \\ 0 \\ -1.0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1.4 \begin{pmatrix} 0.3 \\ 0 \\ 1.1 \\ 0 \\ -0.2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.3 \\ 2.24 \\ -0.7 \\ -0.28 \\ 1.4 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} -14.5 \\ 0.2 \\ 9.9 \\ 15.0 \end{pmatrix}$ to be the image, we see that it is in the set of all

solutions.

When a linear map is “onto”

From secondary-school functions, you know that the function $y = x^2 + 1$ maps the reals to the reals, but there is no x value such that, for example, $y = -2$. This function does not map the reals onto the reals. On the other hand, you know the function $y = x(x - 1)(x + 1)$ has at least one value of x that maps onto any value of y . In some cases, there is exactly one, in two cases there are exactly two, and for others there are three. This second function maps the real numbers *onto* the reals.

A linear map $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is *onto* if every vector in the co-domain V has at least one pre-image. There are, in fact, three distinct cases:

1. there are vectors in the co-domain that have no pre-images,
2. every vector in the co-domain has exactly one pre-image, and
3. every vector in the co-domain has infinitely many pre-images.

If $\text{rank}(A) < m$, there are vectors in the co-domain that have no pre-image, and thus A would not be “onto”. If $\text{rank}(A) = m$, then every vector in the co-domain has at least one pre-image, and thus this linear mapping would be described as “onto”. This is equivalent to saying that a linear map is onto if and only if the rank equals the dimension of the co-domain, or a linear map is onto if and only if the rank equals the number of rows of the matrix.

If $n < m$, it is impossible for a linear mapping to be onto.

Example 1: $B: \mathbf{R}^3 \rightarrow \mathbf{R}^4$

For the map B , we have that $3 < 4$, and thus we may immediately conclude that the mapping B is not onto. As it happens, $\text{rank}(B) = 3$.

Example 2: $C: \mathbf{R}^4 \rightarrow \mathbf{R}^5$

For the map C , we have that $4 < 5$, and thus we may immediately conclude that the mapping C is not onto. As it happens, $\text{rank}(C) = 2$.

Example 3: $D: \mathbf{R}^3 \rightarrow \mathbf{R}^3$

For the operator D , the dimension of the domain equals the dimension of the co-domain ($3 = 3$), so we must examine

the rank. From $\begin{pmatrix} 1.2 & -1.2 & -2.1 \\ 4.0 & 3.0 & -2.0 \\ -2.0 & 1.5 & 6.0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}$, we see that $\text{rank}(D) = 3$, which equals the dimension of the

co-domain, and thus this mapping is onto.

Example 4: $E: \mathbf{R}^4 \rightarrow \mathbf{R}^4$

For the operator E , the dimension of the domain equals the dimension of the co-domain ($4 = 4$), so we must examine

the rank. From $\begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 \\ 5.0 & 3.0 & -2.0 & 3.0 \\ -1.5 & 3.1 & -1.4 & 0.1 \\ -0.5 & 2.5 & 0.8 & 3.4 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -2 & 3 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and since the rank of the matrix is 3, which is less

than the dimension of the co-domain, there are vectors in the co-domain that have no pre-image, and thus this linear operator is not onto.

Example 5: $F: \mathbb{R}^5 \rightarrow \mathbb{R}^3$

For the map F , the dimension of the domain is greater than the dimension of the co-domain ($5 > 3$), so we must

examine the rank. From
$$\begin{pmatrix} -0.8 & 2.8 & -1.4 & 1.0 & 0.4 \\ 1.6 & -2.6 & 1.3 & 4.5 & 3.2 \\ 2.0 & -2.0 & 1.0 & 0.0 & 4.0 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 1 & 0 & 4 \\ 0 & 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix},$$
 we see that $\text{rank}(F) = 3$, which

equals the dimension of the co-domain, and thus this mapping is onto.

Example 6: $G: \mathbb{R}^6 \rightarrow \mathbb{R}^4$

For the map G , the dimension of the domain is greater than the dimension of the co-domain ($6 > 4$), so we must

examine the rank. From
$$\begin{pmatrix} -3.5 & -1.4 & 4.1 & 0.6 & 0.3 & -3.4 & 0 \\ 2.0 & 0.8 & -0.4 & 1.6 & -0.7 & -0.3 & 0 \\ -1.5 & -0.6 & 0.7 & -0.8 & 4.6 & 0.6 & 0 \\ 5.0 & 2.0 & -3.0 & 2.0 & 1.0 & 2.0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & -3 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
 we see

that $\text{rank}(G) = 3$, and this is less than the dimension of the co-domain, and thus there are vectors in the co-domain that have no pre-image, and thus the mapping G is not onto.

When a linear map is “one-to-one”

From secondary-school functions, you know that the function $y = x^2 + 1$ maps the reals to the reals, but there are two values of x that map to the value $y = 2$, namely $x = 1$ and $x = -1$. Thus, this function is not one-to-one. The function $y = x^3$, however, is one-to-one, because if there is a value of x such that $y = x^3$, there is exactly one x value only. In this case, every y value has exactly one pre-image. The exponential function is also one-to-one: if $y > 0$, there is only one value of x such that $y = e^x$.

A linear map is one-to-one if and only if the rank equals the dimension of the domain. Equivalently, a linear map is one-to-one if and only if the rank equals the number of columns in the matrix. Equivalently, a linear map is one-to-one if and only if in solving for the pre-image of a vector in the range involves no free variables. If the dimension of the domain is greater than the dimension of the range, the linear operator cannot be one-to-one.

Example 1: $B: \mathbb{R}^3 \rightarrow \mathbb{R}^4$

For the map B , $3 < 4$ and thus we must inspect the rank to determine if this linear map is one-to-one. Looking at

$$\begin{pmatrix} -0.4 & -2.6 & -2.1 \\ -1.4 & -2.9 & -3.7 \\ 2.0 & 3.0 & 1.0 \\ 0.2 & 4.3 & 5.1 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$
 we see that $\text{rank}(B) = 3$ and this equals the dimension of the domain (the

number of columns). Thus, this mapping is one-to-one. You will recall that the pre-image of the zero vector in $\mathbb{R}^4$ is just the zero vector in $\mathbb{R}^3$.

Example 2: $C: \mathbb{R}^4 \rightarrow \mathbb{R}^5$

For the map C , $4 < 5$ and thus we must inspect the rank to determine if this linear map is one-to-one. Looking at

$$\begin{pmatrix} -1.0 & -0.4 & -2.0 & -2.6 \\ 5.0 & 2.0 & 4.0 & -2.0 \\ 3.0 & 1.2 & 0.8 & -5.2 \\ -2.5 & -1.0 & 0.0 & 6.0 \\ 2.0 & 0.8 & 2.0 & 0.2 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 4 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
 we see $\text{rank}(C) = 2 < 4$, and thus this linear operator is not one-to-

one. You will recall that the pre-image of the zero vector in $\mathbb{R}^5$ was a two-dimensional subspace of $\mathbb{R}^4$.

Example 3: $D: \mathbf{R}^3 \rightarrow \mathbf{R}^3$

For the operator D , $3 = 3$, so we must inspect the rank to determine if this linear operator is one-to-one. Looking at

$$\begin{pmatrix} 1.2 & -1.2 & -2.1 \\ 4.0 & 3.0 & -2.0 \\ -2.0 & 1.5 & 6.0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}, \text{ we see that } \text{rank}(D) = 3 \text{ which equals the dimension of the domain (that is,}$$

the number of columns). Thus, this linear operator is one-to-one. You will also recall that the pre-image of the zero vector in $\mathbf{R}^3$ is just the zero vector in $\mathbf{R}^3$.

Example 4: $E: \mathbf{R}^4 \rightarrow \mathbf{R}^4$

For the operator E , $4 = 4$ so we must inspect the rank to determine if this linear operator is one-to-one. Looking at

$$\begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 \\ 5.0 & 3.0 & -2.0 & 3.0 \\ -1.5 & 3.1 & -1.4 & 0.1 \\ -0.5 & 2.5 & 0.8 & 3.4 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -2 & 3 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ we see that } \text{rank}(E) = 3 \text{ which is less than the dimension of the}$$

domain (in this case, equal to 4), and thus this linear operator is not one-to-one. You will recall that the pre-image of the zero vector was a 1-dimensional subspace of $\mathbf{R}^4$.

Example 5: $F: \mathbf{R}^5 \rightarrow \mathbf{R}^3$

For the map F , $5 > 3$, so this linear mapping cannot be one-to-one.

Example 6: $G: \mathbf{R}^6 \rightarrow \mathbf{R}^4$

For the map G , $6 > 4$, so this linear mapping cannot be one-to-one.

When a linear map is “one-to-one and onto”

A linear map $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ (represented by an $m \times n$ matrix) cannot be onto if $n < m$ and the linear map cannot be one-to-one if $n > m$. Thus, the only possibility for a linear map being one-to-one and onto simultaneously is if $m = n$; that is, if A is a linear operator.

A linear operator $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is one-to-one and onto if and only if $\text{rank}(A) = n$. Equivalently, a linear operator is one-to-one and onto if and only if the pre-image of the zero vector $\mathbf{0}_n$ is just the zero vector, and no more.

Visually, a linear operator is one-to-one and onto if and only if the row-equivalent row-echelon form looks like matrices such as

$$\begin{pmatrix} \blacksquare & & \\ & \blacksquare & \\ & & \blacksquare \end{pmatrix} \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare & \\ & \blacksquare & \blacksquare & \blacksquare \\ & & \blacksquare & \blacksquare & \blacksquare \\ & & & \blacksquare & \blacksquare \end{pmatrix} \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \\ & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & & \blacksquare & \blacksquare & \blacksquare \\ & & & & \blacksquare & \blacksquare \\ & & & & & \blacksquare \end{pmatrix}$$

Note that a finite-dimensional linear operator that is not onto is also not one-to-one, and vice versa.

Example 3: $D: \mathbf{R}^3 \rightarrow \mathbf{R}^3$

For the linear operator D , we see from $\begin{pmatrix} 1.2 & -1.2 & -2.1 \\ 4.0 & 3.0 & -2.0 \\ -2.0 & 1.5 & 6.0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ that $\text{rank}(D) = 3$ which equals the

dimension of the vector space (both domain and co-domain). Thus, this linear operator is both one-to-one and onto: for each vector in $\mathbf{R}^3$, there is exactly one unique pre-image of that vector.

Example 4: $E: \mathbf{R}^4 \rightarrow \mathbf{R}^4$

For the operator E , we see from $\begin{pmatrix} 1.0 & -0.2 & 1.0 & 1.9 \\ 5.0 & 3.0 & -2.0 & 3.0 \\ -1.5 & 3.1 & -1.4 & 0.1 \\ -0.5 & 2.5 & 0.8 & 3.4 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -2 & 3 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ that $\text{rank}(E) = 3 < 4$, with 4 being

the dimension of the vector space (both the domain and co-domain). Thus, this linear operator is neither one-to-one nor is it onto. This means that

1. there are vectors in $\mathbf{R}^4$ that have no pre-image, and
2. if a vector in $\mathbf{R}^4$ has a pre-image, then it has infinitely many pre-images.

We have already seen these above.

Summary of one-to-one and onto

A summary of the possibilities are:

Condition	Visual examples	One-to-one	Onto
$n > m$		Never one-to-one	Onto if $\text{rank}(A) = m$
$m = n$		One-to-one and onto if and only if $\text{rank}(A) = m = n$	
$n < m$		One-to-one if $\text{rank}(A) = n$	Never onto

Other thoughts

Every vector in $\mathbf{R}^m$ has a pre-image under a mapping A if and only if $\text{rank}(A) = m$. If every vector in $\mathbf{R}^m$ has a pre-image under A , we say that A is *onto*. A linear mapping A is onto if and only if $(A \parallel \mathbf{v})$ has at least one solution for every vector $\mathbf{v} \in \mathbf{R}^m$.

If $\text{rank}(A) < m$, there are infinitely many vectors in $\mathbf{R}^m$ that have no pre-image.

If $m > n$, a linear mapping A can never be onto. That is, there will always be vectors in $\mathbf{R}^m$ that have no pre-image in $\mathbf{R}^n$.

Given a linear mapping A , if a vector in $\mathbf{R}^m$ that has a pre-image in $\mathbf{R}^n$ also has a unique pre-image, we say that A is one-to-one. A mapping A is one-to-one if and only if $\text{rank}(A) = n$. You can see this because solving for $(A \parallel \mathbf{v})$ has a unique solution (if one exists) if and only if $\text{rank}(A) = n$.

If $n > m$, a linear mapping A can never be one-to-one. That is, if a vector $\mathbf{v} \in \mathbf{R}^m$ has a pre-image (though there is no guarantee a pre-image exists), it must have infinitely many vectors.

If $\text{rank}(A) < n$, every pre-image that exists has infinitely many vectors in $\mathbf{R}^n$. You may see this because in solving $(A \vdots \mathbf{v})$, there is at least one free variable and so there must be infinitely many solutions to $A\mathbf{u} = \mathbf{v}$.

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are both pre-images of $\mathbf{v}$ under a linear mapping A , then so are all affine combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$, meaning $\alpha\mathbf{u}_1 + (1-\alpha)\mathbf{u}_2$, are also pre-images of $\mathbf{v}$, as

$$\begin{aligned} A(\alpha\mathbf{u}_1 + (1-\alpha)\mathbf{u}_2) &= A(\alpha\mathbf{u}_1) + A((1-\alpha)\mathbf{u}_2) \\ &= \alpha A\mathbf{u}_1 + (1-\alpha)A\mathbf{u}_2 \\ &= \alpha\mathbf{v} + (1-\alpha)\mathbf{v} \\ &= (\alpha + (1-\alpha))\mathbf{v} \\ &= 1 \cdot \mathbf{v} \\ &= \mathbf{v} \end{aligned}$$

Thus, if a vector $\mathbf{v} \in \mathbf{R}^m$ has at least two pre-images, it also has infinitely many pre-images.

If $\mathbf{u}_1, \dots, \mathbf{u}_k$ form a basis for $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then $\text{span}\{A\mathbf{u}_1, \dots, A\mathbf{u}_k\}$ is the image of $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$. If the linear mapping A is one-to-one, then $A\mathbf{u}_1, \dots, A\mathbf{u}_k$ must be linearly independent and thus $A\mathbf{u}_1, \dots, A\mathbf{u}_k$ is also a basis for the image of $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$.