

An $m \times n$ matrix A is a mapping $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ and is represented by

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}.$$

For example, we will consider the following seven matrices

$$B: \mathbf{R}^2 \rightarrow \mathbf{R}^3$$

$$B = \begin{pmatrix} 1.2 & -2.2 \\ -1.8 & 0.1 \\ 2.0 & 3.0 \end{pmatrix}$$

$$D: \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

$$D = \begin{pmatrix} -2.4 & -6.8 \\ 4.0 & 3.0 \end{pmatrix}$$

$$G: \mathbf{R}^3 \rightarrow \mathbf{R}^2$$

$$G = \begin{pmatrix} 1.6 & 0.4 & -0.2 \\ 2.0 & -2.0 & 1.0 \end{pmatrix}$$

$$C: \mathbf{R}^2 \rightarrow \mathbf{R}^3$$

$$C = \begin{pmatrix} -1.0 & -0.4 \\ 5.0 & 2.0 \\ 3.0 & 1.2 \end{pmatrix}$$

$$E: \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

$$E = \begin{pmatrix} 3.5 & 2.1 \\ 5.0 & 3.0 \end{pmatrix}$$

$$H: \mathbf{R}^3 \rightarrow \mathbf{R}^2$$

$$H = \begin{pmatrix} 4.5 & 1.8 & -2.7 \\ 5.0 & 2.0 & -3.0 \end{pmatrix}$$

$$F: \mathbf{R}^3 \rightarrow \mathbf{R}^3$$

$$F = \begin{pmatrix} 0.6 & -2.4 & 1.8 \\ 2.0 & -3.0 & 4.0 \\ -0.8 & 6.2 & -3.6 \end{pmatrix}$$

If $m = n$, the matrix is said to be a *linear operator*, otherwise the matrix is said to be a *linear map*. The matrix-vector product $A\mathbf{u}$ is defined as the linear combination of the columns of A with the coefficients taken from the vector $\mathbf{u}$; that is, if

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$$

then

$$A\mathbf{u} = u_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + u_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ \vdots \\ a_{m,2} \end{pmatrix} + u_3 \begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \\ \vdots \\ a_{m,3} \end{pmatrix} + \cdots + u_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \\ \vdots \\ a_{m,n} \end{pmatrix}.$$

The *image* of a vector $\mathbf{u} \in \mathbf{R}^n$ under the mapping A is that vector $A\mathbf{u} \in \mathbf{R}^m$. If $\mathbf{v} \in \mathbf{R}^m$ and there is a vector $\mathbf{u} \in \mathbf{R}^n$ such that $A\mathbf{u} = \mathbf{v}$, we say that $\mathbf{u}$ is a *pre-image* of $\mathbf{v}$. The pre-image of $\mathbf{v}$ is the set of all vectors in $\mathbf{u} \in \mathbf{R}^n$ such that $A\mathbf{u} = \mathbf{v}$. We can find the pre-image of a vector $\mathbf{v}$ by finding all solutions to

$$(A \vdots \mathbf{v}) = \left(\begin{array}{ccccc|c} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & v_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & v_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} & v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} & v_m \end{array} \right).$$

For our seven examples, the pre-images of the vectors we have found will, in some cases, be just that vector, and in others, will be a set of vectors containing the vector in question. We will find bases for both the range and null spaces of each of these linear maps or operators.

Example 1: $B: \mathbf{R}^2 \rightarrow \mathbf{R}^3$

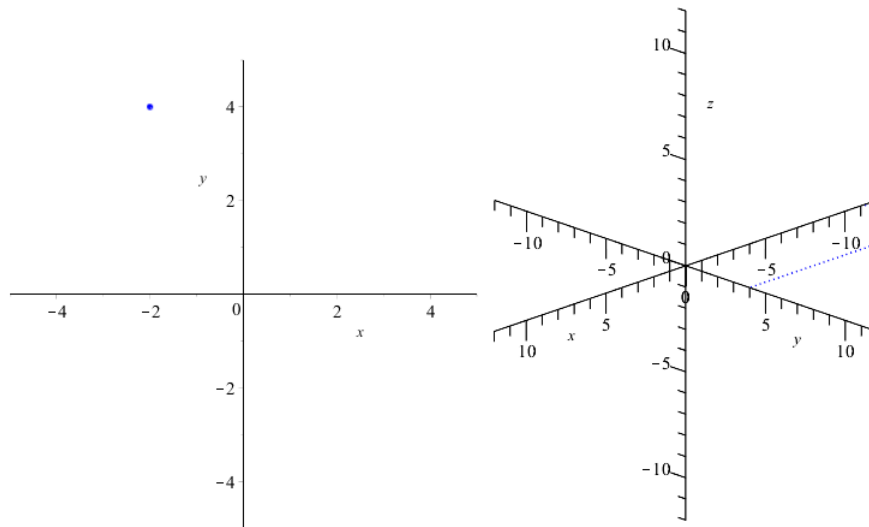
Given the linear mapping $B: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ where $B = \begin{pmatrix} 1.2 & -2.2 \\ -1.8 & 0.1 \\ 2.0 & 3.0 \end{pmatrix}$ is a 3×2 matrix, the dimension of the domain is 2

and the dimension of the co-domain is 3. Because the dimension of the domain is smaller than the dimension of the co-domain, the mapping can never be onto—there will always be vectors in the co-domain that have no pre-image.

Image of a vector under B

If $\mathbf{u} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \in \mathbf{R}^2$, $B\mathbf{u} = (-2) \begin{pmatrix} 1.2 \\ -1.8 \\ 2.0 \end{pmatrix} + 4 \begin{pmatrix} -2.2 \\ 0.1 \\ 3.0 \end{pmatrix} = \begin{pmatrix} -11.2 \\ 4.0 \\ 8.0 \end{pmatrix} \in \mathbf{R}^3$, and thus, the image of $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ under B is $\begin{pmatrix} -11.2 \\ 4.0 \\ 8.0 \end{pmatrix}$,

as shown in the following two graphs:



A pre-image of the vector $\begin{pmatrix} -11.2 \\ 4.0 \\ 8.0 \end{pmatrix}$ is the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$. Under certain circumstances, the pre-image will be unique.

A basis for the range of B

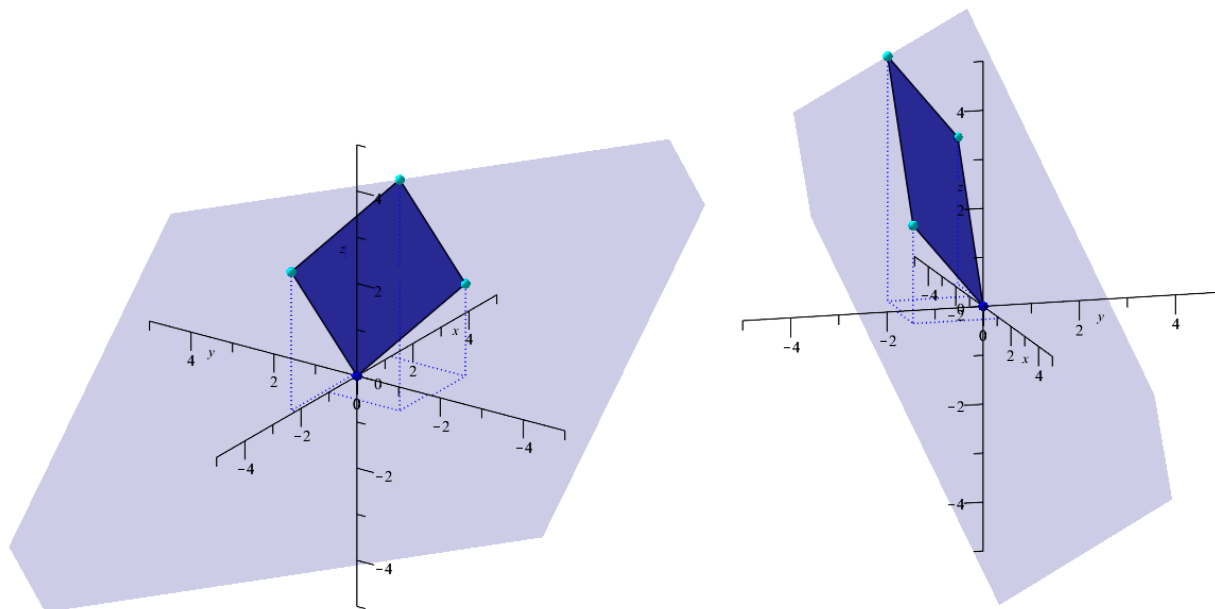
We will now find a basis for the range; that is, all vectors in $\mathbf{R}^3$ that have a pre-image under B will be in the span of this basis.

For the map B , we have $\begin{pmatrix} 1.2 & -2.2 \\ -1.8 & 0.1 \\ 2.0 & 3.0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 \\ 0 & -4 \\ 0 & 0 \end{pmatrix}$, and thus the dimension of the range is 2 (being the rank of the

matrix B), and a basis for the range are those columns in B that correspond to columns with leading non-zero entries in the row-equivalent matrix in row-echelon form, thus the basis for the range is

$$\begin{pmatrix} 1.2 \\ -1.8 \\ 2.0 \end{pmatrix}, \begin{pmatrix} -2.2 \\ 0.1 \\ 3.0 \end{pmatrix}.$$

Because the dimension of the range is less than the dimension of the co-domain, there are vectors in the co-domain that have no pre-image. Thus, the mapping is not “onto”. The only vectors in the co-domain that are mapped onto are shown in the following graphs, these being different perspectives of the same plane in $\mathbf{R}^3$. No point above or below this plane has a pre-image.

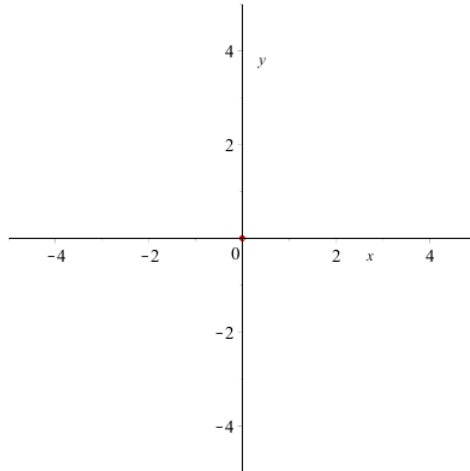


A basis for the null space of B

We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^2$ that will be all vectors in $\mathbf{R}^2$ that are mapped by B onto the zero vector $\mathbf{0}_3$.

For the map B , we have $\left(\begin{array}{cc|c} 1.2 & -2.2 & 0 \\ -1.8 & 0.1 & 0 \\ 2.0 & 3.0 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{array} \right)$ and thus we note because the rank equals the number of

columns, a unique solution exists, as $-4\alpha_2 = 0$, and thus $\alpha_2 = 0$, and substituting this into the first equation we have that $2\alpha_1 = 0$, so $\alpha_1 = 0$. Therefore, as we already know that the zero-vector in $\mathbf{R}^n$ maps onto the zero-vector of $\mathbf{R}^m$, this must be the unique vector in the null space. Because the null space is the zero vector, the mapping is “one-to-one”. Note that the zero vector is a subspace of $\mathbf{R}^n$. Visually, given to domain $\mathbf{R}^2$, the null space is simply the origin (the zero vector).



Thus, every point in $\mathbf{R}^2$ is mapped to a unique point on the subspace of $\mathbf{R}^3$.

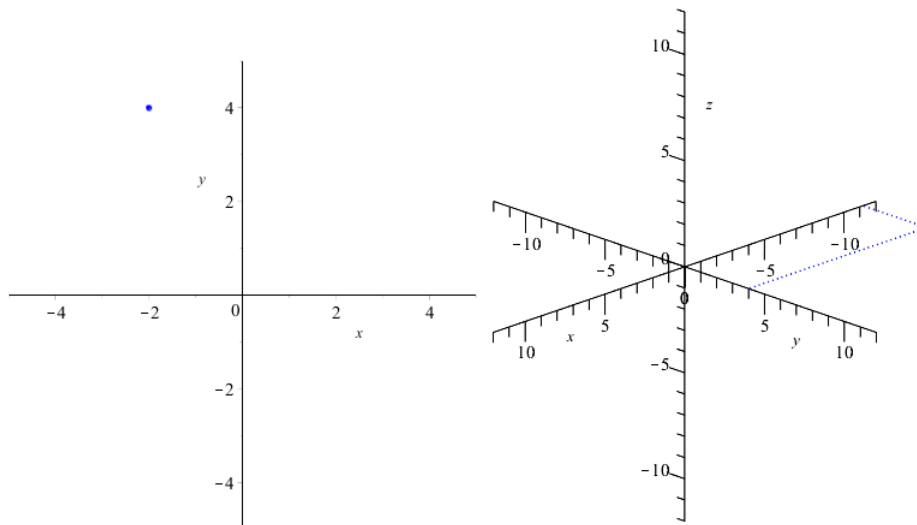
Finding all pre-images of a given vector

For the map B , given $\mathbf{v} = \begin{pmatrix} -11.2 \\ 4.0 \\ 8.0 \end{pmatrix}$, we have that $\begin{pmatrix} 1.2 & -2.2 & -11.2 \\ -1.8 & 0.1 & 4.0 \\ 2.0 & 3.0 & 8.0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 8 \\ 0 & -4 & -16 \\ 0 & 0 & 0 \end{pmatrix}$ and thus we note

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, and
2. the rank of the matrix equals the number of columns, so a unique solution exists.

Therefore, we can find this unique pre-image of $\mathbf{v}$ using backward substitution; that is, $\mathbf{u} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. Thus, there is

only one vector in $\mathbf{R}^2$ that is mapped onto $\mathbf{v} = \begin{pmatrix} -11.2 \\ 4.0 \\ 8.0 \end{pmatrix}$ in $\mathbf{R}^3$.



If, however, we attempt to find a pre-image for $\mathbf{v} = \begin{pmatrix} -11.2 \\ 7.0 \\ 8.0 \end{pmatrix}$, we see that $\left(\begin{array}{cc|c} 1.2 & -2.2 & -11.2 \\ -1.8 & 0.1 & 7.0 \\ 2.0 & 3.0 & 8.0 \end{array} \right) \sim \left(\begin{array}{cc|c} 2 & 3 & 8 \\ 0 & -4 & -16 \\ 0 & 0 & 3 \end{array} \right),$

and thus no solution exists as the rank of the matrix is 2, while the rank of the augmented matrix is 3, and thus this vector $\mathbf{v}$ has no pre-image.

Properties of B

The mapping B is one-to-one (every vector in $\mathbf{R}^3$ that has a pre-image has a unique pre-image), but not onto (not every vector in $\mathbf{R}^3$ has a pre-image).

Example 2: $C: \mathbf{R}^2 \rightarrow \mathbf{R}^3$

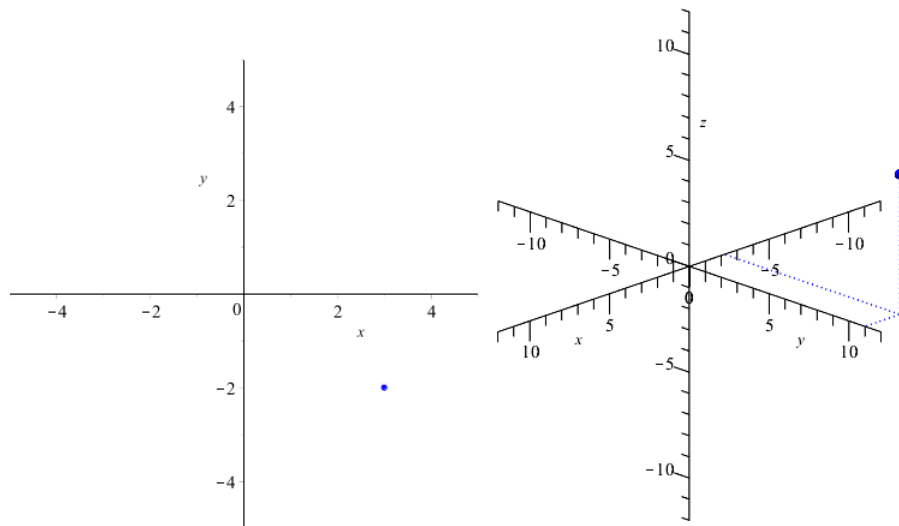
Given the linear mapping $C: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ where $C = \begin{pmatrix} -1.0 & -0.4 \\ 5.0 & 2.0 \\ 3.0 & 1.2 \end{pmatrix}$ is a 3×2 matrix, the dimension of the domain is 2

and the dimension of the co-domain is 3. Because the dimension of the domain is smaller than the dimension of the co-domain, the mapping can never be onto—there will always be vectors in the co-domain that have no pre-image.

Image of a vector under C

If $\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \in \mathbf{R}^2$, $C\mathbf{u} = 3 \begin{pmatrix} -1.0 \\ 5.0 \\ 3.0 \end{pmatrix} + (-2) \begin{pmatrix} -0.4 \\ 2.0 \\ 1.2 \end{pmatrix} = \begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix} \in \mathbf{R}^3$, and thus the image of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ under C is $\begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}$, as

shown in the following two graphs:



A pre-image of the vector $\begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}$ is the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Under certain circumstances, the pre-image will be unique.

A basis for the range of C

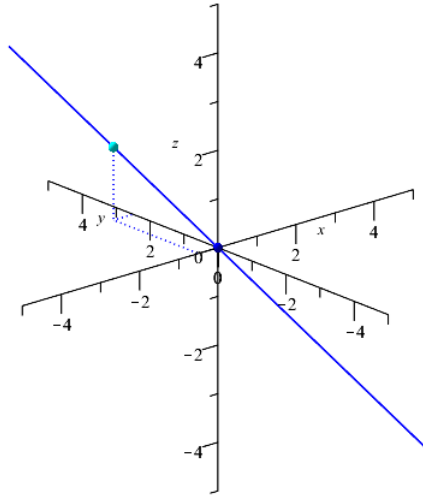
We will now find a basis for the range; that is, all vectors in $\mathbf{R}^3$ that have a pre-image under C will be in the span of this basis.

For the map C , we have $\begin{pmatrix} -1.0 & -0.4 \\ 5.0 & 2.0 \\ 3.0 & 1.2 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, and thus the dimension of the range is 1 (being the rank of the

matrix C), and a basis for the range is

$$\begin{pmatrix} -1.0 \\ 5.0 \\ 3.0 \end{pmatrix}.$$

Because the dimension of the range (1) is less than the dimension of the co-domain (3), there are vectors in the co-domain that have no pre-image. In this case, the range forms a line in $\mathbf{R}^3$. The following graph shows how this line is a subspace of $\mathbf{R}^3$. Any point not on this line does not have a pre-image under this mapping.



A basis for the null space of C

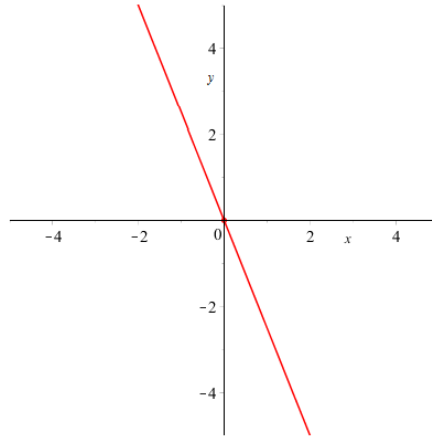
We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^2$ that will be all vectors in $\mathbf{R}^2$ that are mapped by C onto the zero vector $\mathbf{0}_3$.

For the map C , we have $\begin{pmatrix} -1.0 & -0.4 & 0 \\ 5.0 & 2.0 & 0 \\ 3.0 & 1.2 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In this example, because the rank of the matrix is less

than the number of columns, there are an infinitely many number of vectors that map onto the zero vector $\mathbf{0}_3$. Put another way, because $\dim(\mathbf{R}^2) - \text{rank}(C) = 2 - 1 = 1$, the dimension of the null space is 1. We can find all these solutions using backward substitution: $\alpha_1 = -0.4\alpha_2$, so the collection of all solutions is of the form

$$\alpha_2 \begin{pmatrix} -0.4 \\ 1 \end{pmatrix}.$$

You may note that all scalar multiples of this vector forms a subspace of $\mathbf{R}^2$. This null space is shown in this graph:



Every point on this line is mapped to $\mathbf{0}_3$, while every point off of this line is mapped to a non-zero vector in $\mathbf{R}^3$.

For example, you will note that the image of the vector

$$-7.2 \begin{pmatrix} -0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.88 \\ -7.2 \end{pmatrix}$$

is the zero vector.

Finding all pre-images of a given vector

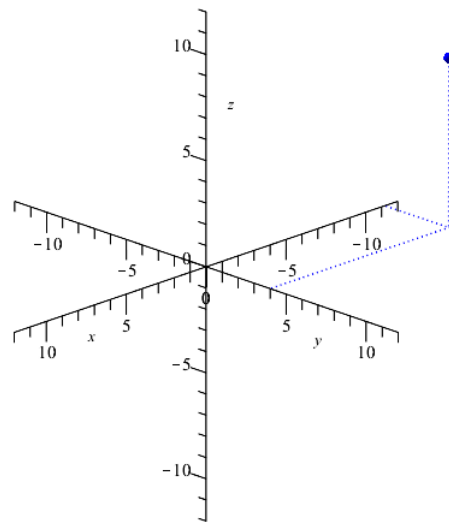
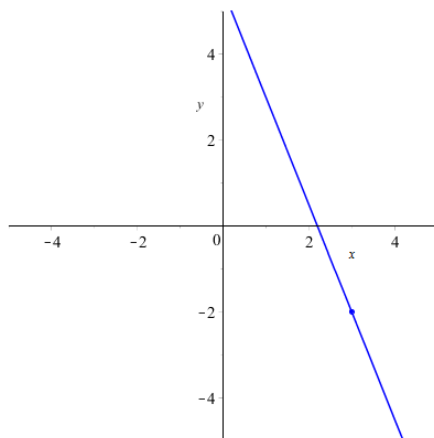
For the map C , given $\mathbf{v} = \begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}$, we have that $\begin{pmatrix} -1.0 & -0.4 & -2.2 \\ 5.0 & 2.0 & 11.0 \\ 3.0 & 1.2 & 6.6 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and thus we note

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, and
2. the rank of the matrix is less than the number of columns, so infinitely many solutions exist.

We can find all solutions by assigning a free variable to Column 2, and thus $\alpha_1 = 2.2 - 0.4\alpha_2$, and thus all solutions are in the form

$$\begin{pmatrix} 2.2 - 0.4\alpha_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2.2 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -0.4 \\ 1 \end{pmatrix}$$

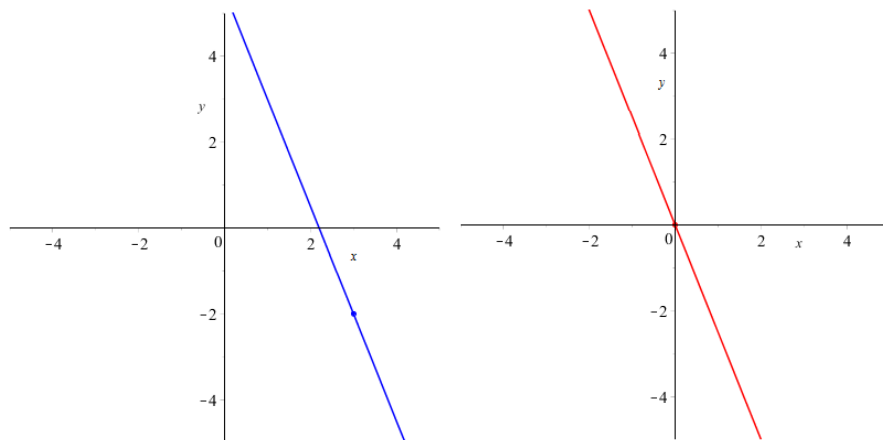
Thus, all of the points that map onto $\mathbf{v} = \begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}$, including $\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \in \mathbf{R}^2$.



If you try it out, you will see that if $\mathbf{u} = \begin{pmatrix} 2.2 \\ 0 \end{pmatrix} + 12.7 \begin{pmatrix} -0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2.88 \\ 12.70 \end{pmatrix}$, then

$$C\mathbf{u} = \begin{pmatrix} -1.0 & -0.4 \\ 5.0 & 2.0 \\ 3.0 & 1.2 \end{pmatrix} \begin{pmatrix} -2.88 \\ 12.7 \end{pmatrix} = \begin{pmatrix} 2.88 \\ -14.4 \\ -8.64 \end{pmatrix} + \begin{pmatrix} -5.08 \\ 25.4 \\ 15.24 \end{pmatrix} = \begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}.$$

Compare all solutions to $C\mathbf{u} = \begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}$ and the null space of C (all solutions to $C\mathbf{u} = \mathbf{0}_3$):



You will see that each point that is a solution to $C\mathbf{u} = \begin{pmatrix} -2.2 \\ 11.0 \\ 6.6 \end{pmatrix}$ is one pre-image plus a vector in the null space.

If, however, we attempt to find a pre-image for $\mathbf{v} = \begin{pmatrix} -11.2 \\ 7.0 \\ 8.0 \end{pmatrix}$, we see that $\left(\begin{array}{cc|c} 1.2 & -2.2 & -11.2 \\ -1.8 & 0.1 & 7.0 \\ 2.0 & 3.0 & 8.0 \end{array} \right) \sim \left(\begin{array}{cc|c} 2 & 3 & 8 \\ 0 & -4 & -16 \\ 0 & 0 & 3 \end{array} \right),$

and thus no solution exists as the rank of the matrix is 2, while the rank of the augmented matrix is 3, and thus this vector $\mathbf{v}$ has no pre-image. Thus, the mapping is not “onto”.

Properties of C

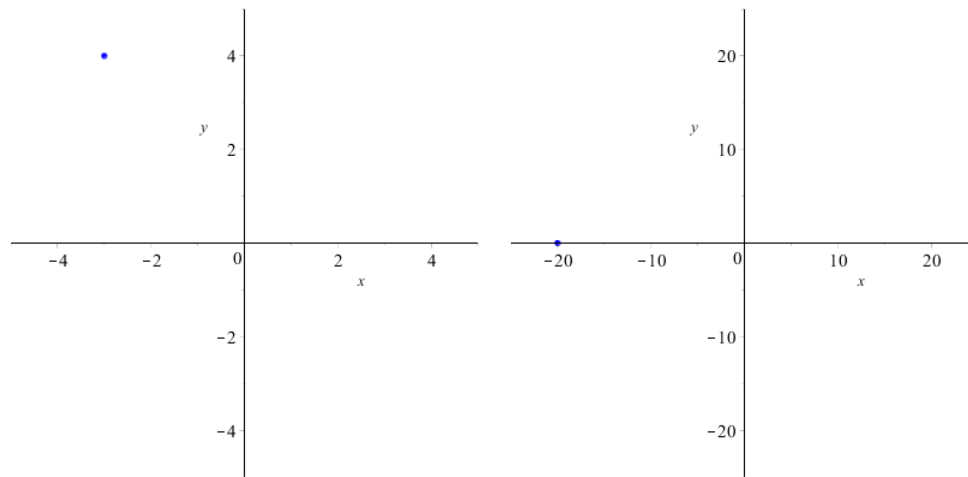
The mapping C is not one-to-one (every vector in $\mathbf{R}^3$ that has a pre-image has multiple pre-images), and not onto (not every vector in $\mathbf{R}^3$ has a pre-image).

Example 3: $D:\mathbf{R}^2 \rightarrow \mathbf{R}^2$

Given the linear operator $D:\mathbf{R}^2 \rightarrow \mathbf{R}^2$ where $D = \begin{pmatrix} -2.4 & -6.8 \\ 4.0 & 3.0 \end{pmatrix}$ is a 2 x 2 matrix, the dimension of the domain is 2 and the dimension of the co-domain is 2. Because the dimension of the domain equals the dimension of the co-domain, the operator will either be both one-to-one and onto, or neither.

Image of a vector under D

If $\mathbf{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \in \mathbf{R}^2$, $D\mathbf{u} = (-3)\begin{pmatrix} -2.4 \\ 4.0 \end{pmatrix} + 4\begin{pmatrix} -6.8 \\ 3.0 \end{pmatrix} = \begin{pmatrix} -20.0 \\ 0.0 \end{pmatrix} \in \mathbf{R}^2$, and thus the image of $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ under D is $\begin{pmatrix} -20.0 \\ 0.0 \end{pmatrix}$, as shown in the following two graphs:



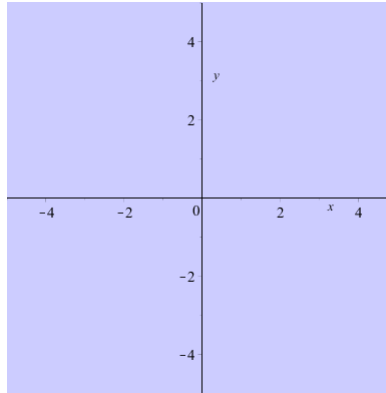
A pre-image of the vector $\begin{pmatrix} -20.0 \\ 0.0 \end{pmatrix}$ is the vector $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$. Under certain circumstances, the pre-image will be unique.

A basis for the range of D

We will now find a basis for the range; that is, all vectors in $\mathbf{R}^2$ that have a pre-image under D will be in the span of this basis. For the operator D , we have $\begin{pmatrix} -2.4 & -6.8 \\ 4.0 & 3.0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 \\ 0 & -5 \end{pmatrix}$ and thus, and thus the dimension of the range is 2 (being the rank of the matrix D), and a basis for this range are the two vectors

$$\begin{pmatrix} -2.4 \\ 4.0 \end{pmatrix}, \begin{pmatrix} -6.8 \\ 3.0 \end{pmatrix}.$$

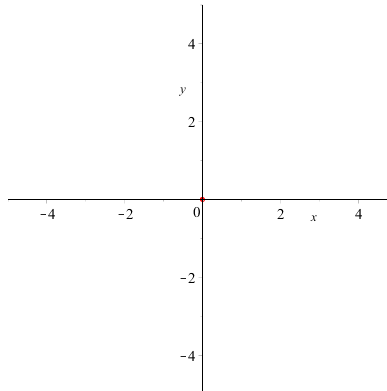
Because the dimension of the co-domain is equal to the dimension of the range, this means that every vector in the co-domain has a pre-image. In addition, as each pre-image is unique, this indicates that every vector in the domain has one image, and each vector in the co-domain has exactly one pre-image. Graphically, the range is all of $\mathbf{R}^2$, so every point in $\mathbf{R}^2$ has a pre-image.



A basis for the null space of D

We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^2$ that will be all vectors in $\mathbf{R}^2$ that are mapped by D onto the zero vector $\mathbf{0}_2$.

For the map D , we have $\begin{pmatrix} -2.4 & -6.8 & 0 \\ 4.0 & 3.0 & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 0 \\ 0 & -5 & 0 \end{pmatrix}$. In this example, because the rank of the matrix is equal than the number of columns (that is, the rank equals the dimension of the domain, or $\text{rank}(A) = \dim(\mathbf{R}^2)$), there is only one vector that maps onto the zero vector $\mathbf{0}_2$, name $\mathbf{0}_2$. Every other point in the domain is mapped onto a non-zero vector in the range.



Finding all pre-images of a given vector

Because the rank of the matrix equals the dimension of the domain, all vectors that have a pre-image have a unique pre-image. Because the rank of the matrix equals the dimension of the co-domain, all vectors in the co-domain have a pre-image. Thus, the matrix D is one-to-one and onto.

For the map D , given $\mathbf{v} = \begin{pmatrix} -20.0 \\ 0.0 \end{pmatrix}$, we have that $\begin{pmatrix} -2.4 & -6.8 & -20.0 \\ 4.0 & 3.0 & 0.0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 0 \\ 0 & -5 & -20 \end{pmatrix}$ and thus we may solve for the one unique solution $\mathbf{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$. If you attempt to find the pre-image of any other vector in the co-domain,

you will find that there is a unique vector that maps onto that vector. The only solutions to $D\mathbf{u} = \begin{pmatrix} -20.0 \\ 0.0 \end{pmatrix}$ are

$\mathbf{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ plus a vector in the null space, but the null space is the singular vector $\mathbf{0}_2$, so $\mathbf{u}$ is the only solution.

Properties of D

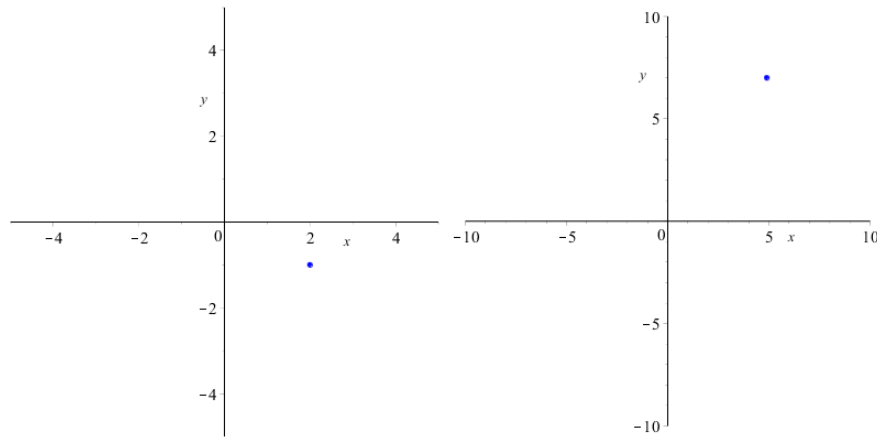
The operator D is both one-to-one (every vector in $\mathbf{R}^2$ that has a pre-image has a unique pre-image) and onto (every vector in the co-domain $\mathbf{R}^2$ has a pre-image). Thus, we could define a linear mapping D^{-1} such that $D^{-1}\mathbf{v} = \mathbf{u}$ whenever $\mathbf{v} = D\mathbf{u}$.

Example 4: $E: \mathbf{R}^2 \rightarrow \mathbf{R}^2$

Given the linear operator $E: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ where $E = \begin{pmatrix} 3.5 & 2.1 \\ 5.0 & 3.0 \end{pmatrix}$ is a 2×2 matrix, the dimension of the domain is 2 and the dimension of the co-domain is 2. Because the dimension of the domain equals the dimension of the co-domain, the operator will either be both one-to-one and onto, or neither.

Image of a vector under E

If $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \mathbf{R}^2$, $E\mathbf{u} = 2 \begin{pmatrix} 3.5 \\ 5.0 \end{pmatrix} + (-1) \begin{pmatrix} 2.1 \\ 3.0 \end{pmatrix} = \begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix} \in \mathbf{R}^2$, and thus the image of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ under E is $\begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$, as shown in the following two graphs:



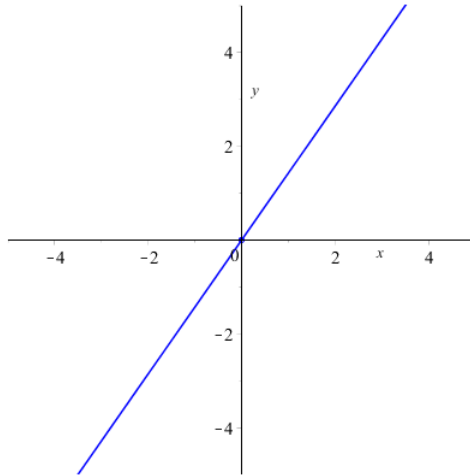
A pre-image of the vector $\begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$ is the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Under certain circumstances, the pre-image will be unique.

A basis for the range of E

We will now find a basis for the range; that is, all vectors in $\mathbf{R}^2$ that have a pre-image under E will be in the span of this basis. For the operator E , we have $\begin{pmatrix} 3.5 & 2.1 \\ 5.0 & 3.0 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix}$ and thus, and thus the dimension of the range is 1 (being the rank of the matrix E), and a basis for this range is the vector in Column 1, the one column in the row-equivalent matrix in row-echelon form with a leading non-zero entry:

$$\begin{pmatrix} 3.5 \\ 5.0 \end{pmatrix}.$$

Because the dimension of the co-domain (2) is greater than the dimension of the range (1), this means that there are vectors in the co-domain that do not have pre-images. Graphically, the range is the span of this vector, so all scalar multiples of this vector, as shown in this graph:



That is, the image of every vector in the domain under E is a point on this line. Any point off of this line does not have a pre-image under the mapping E .

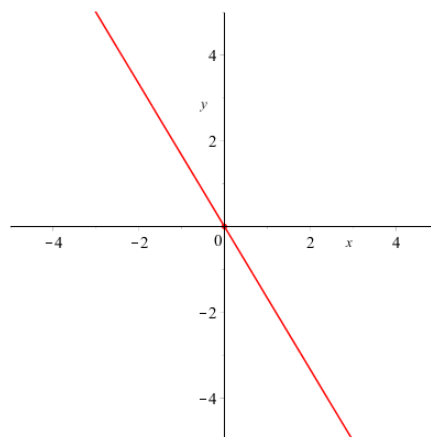
A basis for the null space of E

We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^2$ that will be all vectors in $\mathbf{R}^2$ that are mapped by E onto the zero vector $\mathbf{0}_2$.

For the map E , we have $\left(\begin{array}{cc|c} 3.5 & 2.1 & 0 \\ 5.0 & 3.0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$. In this example, because the rank of the matrix is less than the number of columns (that is, the rank is less than the dimension of the domain, or $\text{rank}(A) < \dim(\mathbf{R}^2)$), there are infinitely many vectors that map onto the zero vector $\mathbf{0}_2$. Solving for this basis, we have $\alpha_1 = -0.6\alpha_2$. Thus, every vector in the null space is of the form

$$\begin{pmatrix} -0.6\alpha_2 \\ \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} -0.6 \\ 1 \end{pmatrix}.$$

Thus, for example, $11.3 \begin{pmatrix} -0.6 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.78 \\ 11.3 \end{pmatrix}$ will map to $\mathbf{0}_2$ under this mapping. This subspace is shown in this graph:



Every other point off of this line is mapped to a non-zero vector in $\mathbf{R}^2$.

Finding all pre-images of a given vector

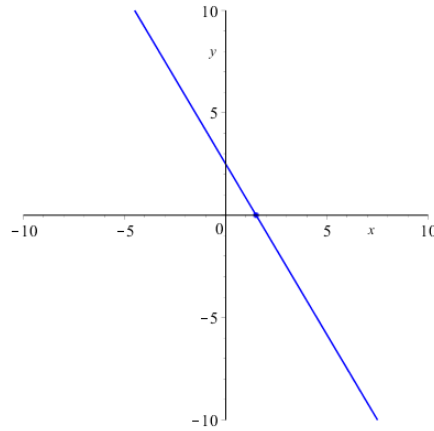
For the operator E , given $\mathbf{v} = \begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$, we have that $\begin{pmatrix} 3.5 & 2.1 & 4.9 \\ 5.0 & 3.0 & 7.0 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & 7 \\ 0 & 0 & 0 \end{pmatrix}$. From the row-echelon form, we have

1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, but
2. the rank of the matrix is one less than the number of columns, so there is one free variable.

Thus, assigning a free variable to Column 2, as this column does not have a leading non-zero entry, we have that the solutions are $\alpha_1 = 1.4 - 0.6\alpha_2$, so the collection of all solutions is of the form

$$\begin{pmatrix} 1.4 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -0.6 \\ 1 \end{pmatrix},$$

as shown in the following graphic, where every point on this line is mapped to $\mathbf{v} = \begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$:



This does not form a subspace in $\mathbf{R}^2$. You will note, however, that

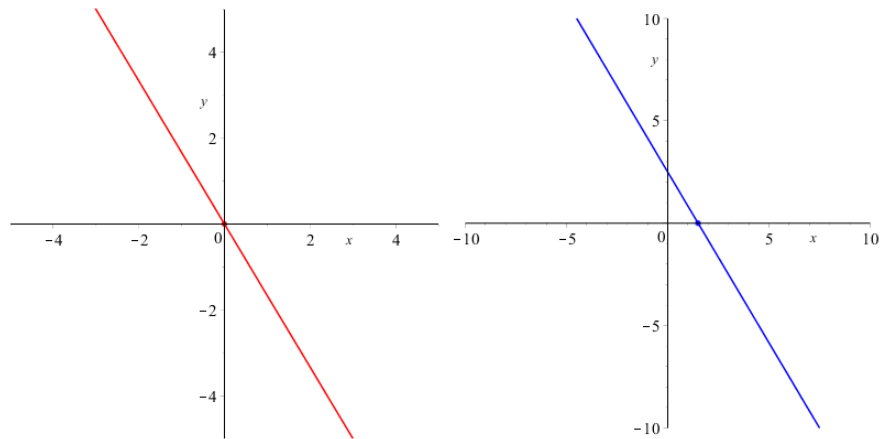
$$\begin{pmatrix} 1.4 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} -0.6 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$ to be the image, we see that it is in the set of all solutions.

Similarly, if you were to find the vector $\mathbf{u} = \begin{pmatrix} 1.4 \\ 0 \end{pmatrix} + (-7.3) \begin{pmatrix} -0.6 \\ 1 \end{pmatrix} = \begin{pmatrix} 5.78 \\ -7.3 \end{pmatrix}$,

you would note that $E\mathbf{u} = \begin{pmatrix} 3.5 & 2.1 \\ 5.0 & 3.0 \end{pmatrix} \begin{pmatrix} 5.78 \\ -7.3 \end{pmatrix} = \begin{pmatrix} 20.23 \\ 28.9 \end{pmatrix} + \begin{pmatrix} -15.33 \\ -21.9 \end{pmatrix} = \begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$, as expected.

You can compare the null space and the solutions to $E\mathbf{u} = \begin{pmatrix} 4.9 \\ 7.0 \end{pmatrix}$ here:



If, however, we attempt to find the pre-image for $\mathbf{v} = \begin{pmatrix} 1.9 \\ 7.0 \end{pmatrix}$, we have

$$\left(\begin{array}{cc|c} 3.5 & 2.1 & 1.9 \\ 5.0 & 3.0 & 7.0 \end{array} \right) \sim \left(\begin{array}{cc|c} 5 & 3 & 7 \\ 0 & 0 & -3 \end{array} \right),$$

and thus the rank of the matrix is one less than rank of the augmented matrix, so no solution exists.

Properties of E

The operator E is neither one-to-one (every vector in $\mathbf{R}^2$ that has a pre-image has multiple pre-images) nor onto (not every vector in the co-domain $\mathbf{R}^2$ has a pre-image).

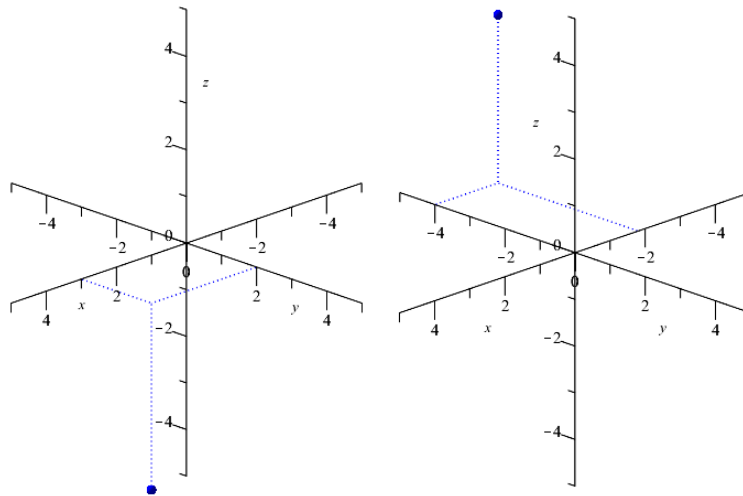
Example 5: $F: \mathbf{R}^3 \rightarrow \mathbf{R}^3$

Given the linear operator $F: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ where $F = \begin{pmatrix} 0.6 & -2.4 & 1.8 \\ 2.0 & -3.0 & 4.0 \\ -0.8 & 6.2 & -3.6 \end{pmatrix}$ is a 3 x 3 matrix, the dimension of the

domain is 3 and the dimension of the co-domain is 3. Because the dimension of the domain equals the dimension of the co-domain, the operator will either be both one-to-one and onto, or neither.

Image of a vector under F

If $\mathbf{u} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \in \mathbf{R}^3$, $F\mathbf{u} = 3 \begin{pmatrix} 0.6 \\ 2.0 \\ -0.8 \end{pmatrix} + (-2) \begin{pmatrix} -1.8 \\ -3.0 \\ 4.2 \end{pmatrix} + (-4) \begin{pmatrix} 1.8 \\ 4.0 \\ -3.6 \end{pmatrix} = \begin{pmatrix} -1.8 \\ -4.0 \\ 3.6 \end{pmatrix} \in \mathbf{R}^3$, and thus the image of $\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$ under F is $\begin{pmatrix} -1.8 \\ -4.0 \\ 3.6 \end{pmatrix}$, as shown in the following two graphs:



A pre-image of the vector $\begin{pmatrix} -1.8 \\ -4.0 \\ 3.6 \end{pmatrix}$ is the vector $\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$. Under certain circumstances, the pre-image will be unique.

A basis for the range of F

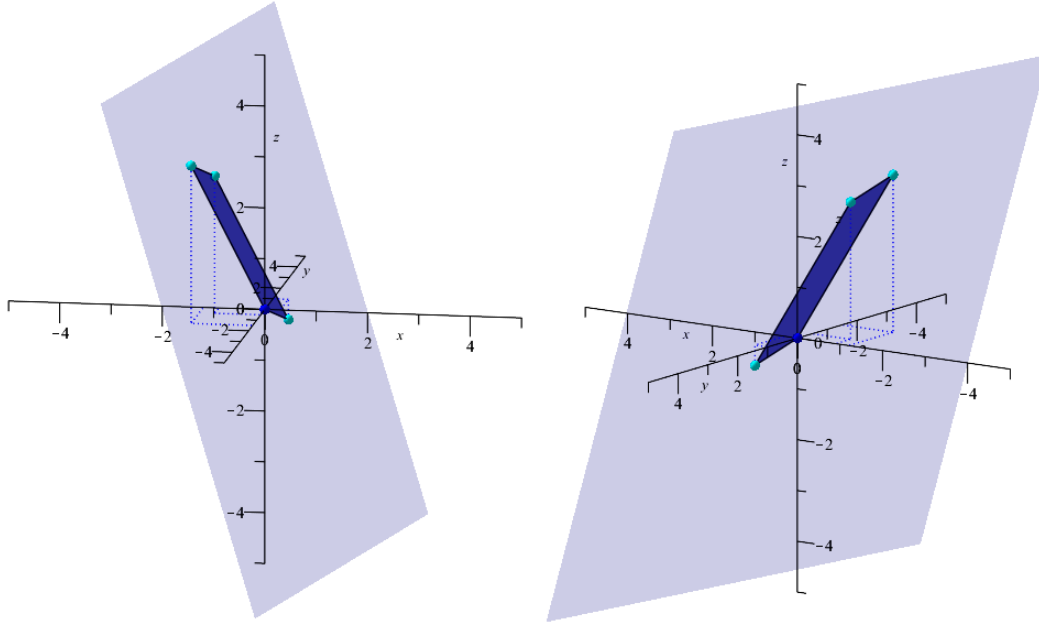
We will now find a basis for the range; that is, all vectors in $\mathbf{R}^3$ that have a pre-image under F will be in the span of

this basis. For the operator F , we have $\begin{pmatrix} 0.6 & -2.4 & 1.8 \\ 2.0 & -3.0 & 4.0 \\ -0.8 & 6.2 & -3.6 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ and thus, and thus the dimension of

the range is 2 (being the rank of the matrix F), and a basis for this range are the vectors in Column 1 and 2, the two columns in the row-equivalent matrix in row-echelon form with a leading non-zero entry:

$$\begin{pmatrix} 0.6 \\ 2.0 \\ -0.8 \end{pmatrix}, \begin{pmatrix} -2.4 \\ -3.0 \\ 6.2 \end{pmatrix}.$$

Because the dimension of the co-domain (3) is greater than the dimension of the range (2), this means that there are vectors in the co-domain that do not have pre-images. Graphically, the range is the span of these two vectors, so including all linear combinations that forms a plane in $\mathbf{R}^3$, as shown in these graphs:



That is, the image of every vector in the domain under F is a point on this plane. Any point off of this plane does not have a pre-image under the mapping F .

A basis for the null space of F

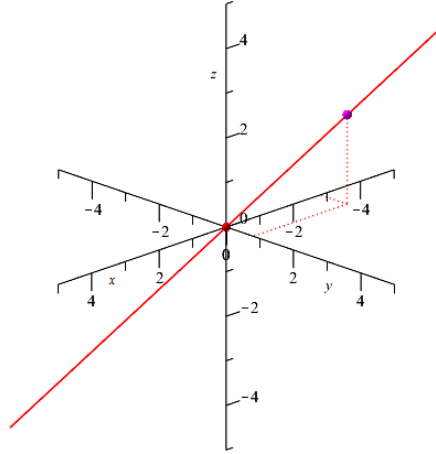
We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^3$ will be all vectors in $\mathbf{R}^3$ that are mapped by F onto the zero vector $\mathbf{0}_3$.

For the map F , we have $\left(\begin{array}{ccc|c} 0.6 & -2.4 & 1.8 & 0 \\ 2.0 & -3.0 & 4.0 & 0 \\ -0.8 & 6.2 & -3.6 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 2 & -3 & 4 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$. In this example, because the rank of the

matrix is less than the number of columns (that is, the rank is less than the dimension of the domain, or $\text{rank}(A) < \dim(\mathbf{R}^2)$), there are infinitely many vectors that map onto the zero vector $\mathbf{0}_3$. Solving for this basis, we have $\alpha_2 = 0.4\alpha_3, \alpha_1 = -1.4\alpha_3$. Thus, every vector in the null space is of the form

$$\begin{pmatrix} -1.4\alpha_3 \\ 0.4\alpha_3 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} -1.4 \\ 0.4 \\ 1 \end{pmatrix}.$$

Thus, for example, $-8.7 \begin{pmatrix} -1.4 \\ 0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 12.18 \\ -3.48 \\ -8.7 \end{pmatrix}$ will map to $\mathbf{0}_3$ under this mapping. This subspace is shown in this graph:



Every other point off of this line is mapped to a non-zero vector in $\mathbf{R}^3$.

Finding all pre-images of a given vector

For the map F , given $\mathbf{v} = \begin{pmatrix} -0.6 \\ -4.0 \\ -0.4 \end{pmatrix}$, we have $\begin{pmatrix} 0.6 & -2.4 & 1.8 & | & -0.6 \\ 2.0 & -3.0 & 4.0 & | & -4.0 \\ -0.8 & 6.2 & -3.6 & | & -0.4 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 4 & | & -4 \\ 0 & 5 & -2 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$. From the row-echelon form, we have,

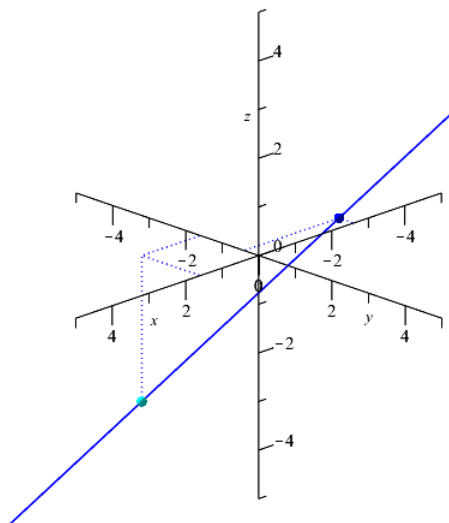
1. the rank of the matrix equals the rank of the augmented matrix, so at least one solution exists, but
2. the rank of the matrix is one less than the number of columns, so there is one free variable.

Thus, assigning the free variable to Column 3, as this column does not have leading non-zero entry, we have that the solutions are $\alpha_2 = -0.4 + 0.4\alpha_3$, $\alpha_1 = -2.6 - 1.4\alpha_3$, so the collection of all solutions is of the form

$$\begin{pmatrix} -2.6 \\ -0.4 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1.4 \\ 0.4 \\ 1 \end{pmatrix},$$

which viewed graphically includes all the points on the line with $\begin{pmatrix} -2.6 \\ -0.4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2.6 \\ -0.4 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} -1.4 \\ 0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.6 \\ -1.6 \\ -3.0 \end{pmatrix}$

highlighted so that you can see approximately how this line appears in $\mathbf{R}^3$:



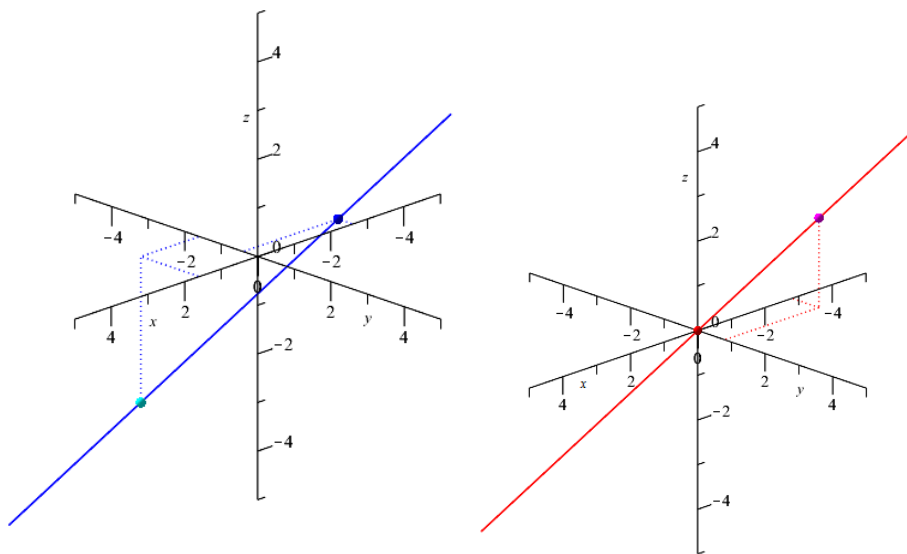
This does not form a subspace in $\mathbf{R}^3$. You will note, however, that

$$\begin{pmatrix} -2.6 \\ -0.4 \\ 0 \end{pmatrix} + (-4) \begin{pmatrix} -1.4 \\ 0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} -0.6 \\ -4.0 \\ -0.4 \end{pmatrix}$ to be the image, we see that it is in the set of all solutions.

You can compare this collection of solutions to $F\mathbf{u} = \begin{pmatrix} -0.6 \\ -4.0 \\ -0.4 \end{pmatrix}$ to the null space here, where you can see that it is

simply the null space shifted:



If, however, we attempt to find the pre-image for $\mathbf{v} = \begin{pmatrix} -2.6 \\ -4.0 \\ -0.4 \end{pmatrix}$, we have

$$\left(\begin{array}{ccc|c} 0.6 & -2.4 & 1.8 & -2.6 \\ 2.0 & -3.0 & 4.0 & -4.0 \\ -0.8 & 6.2 & -3.6 & -0.4 \end{array} \right) \sim \left(\begin{array}{ccc|c} 2 & -3 & 4 & -4 \\ 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & 2 \end{array} \right),$$

and thus the rank of the matrix is one less than rank of the augmented matrix, so no solution exists.

Properties of F

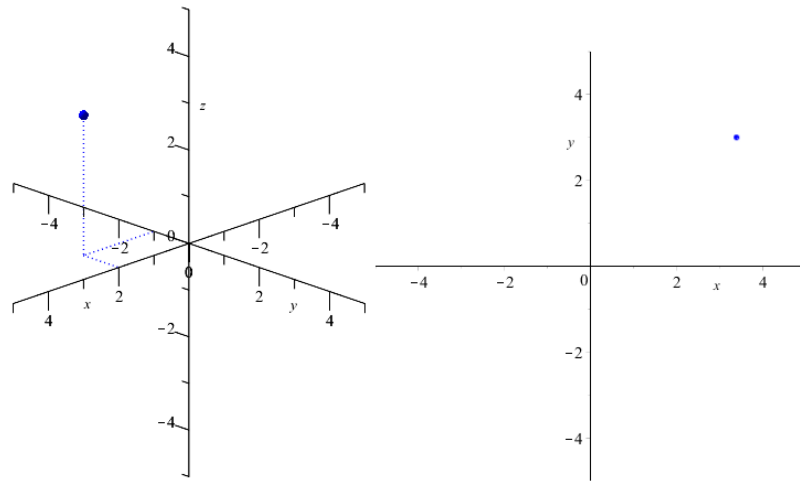
The operator F is neither one-to-one (every vector in $\mathbf{R}^3$ that has a pre-image has multiple pre-images) nor onto (not every vector in the co-domain $\mathbf{R}^3$ has a pre-image).

Example 6: $G: \mathbf{R}^3 \rightarrow \mathbf{R}^2$

Given the linear mapping $G: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ where $G = \begin{pmatrix} 1.6 & 0.4 & -0.2 \\ 2.0 & -2.0 & 1.0 \end{pmatrix}$ is a 2×3 matrix, the dimension of the domain is 3 and the dimension of the co-domain is 2. Because the dimension of the domain is greater than the dimension of the co-domain, the mapping can never be one-to-one—every vector that has a pre-image in the co-domain will have infinitely many pre-images.

Image of a vector under G

If $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \in \mathbf{R}^3$, $G\mathbf{u} = 2 \begin{pmatrix} 1.6 \\ 2.0 \end{pmatrix} + (-1) \begin{pmatrix} 0.4 \\ -2.0 \end{pmatrix} + (-3) \begin{pmatrix} -0.2 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix} \in \mathbf{R}^2$, and thus the image of $\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ under G is $\begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix}$, as shown in the following two graphs:



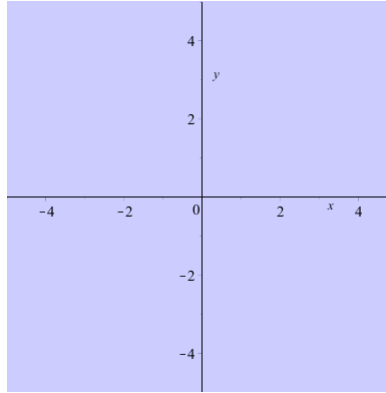
A pre-image of the vector $\begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix}$ is the vector $\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$. As the dimension of the domain is larger than that of the co-domain, there must be infinitely many vectors that map onto $\begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix}$.

A basis for the range of G

We will now find a basis for the range; that is, all vectors in $\mathbf{R}^2$ that have a pre-image under G will be in the span of this basis. For the map G , we have $\begin{pmatrix} 1.6 & 0.4 & -0.2 \\ 2.0 & -2.0 & 1.0 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix}$. From the row-echelon form, we have, that the rank is 2, so because the dimension of the co-domain is 2, this means that each vector in the co-domain has at least one pre-image. Because the rank is less than the number of column vectors, there are infinitely many solutions. From the row-echelon form, a basis for the range are Columns 1 and 2; that is, those columns that have leading non-zero entries in the row-equivalent row-echelon form. Thus, the basis is

$$\begin{pmatrix} 1.6 \\ 2.0 \end{pmatrix}, \begin{pmatrix} 0.4 \\ -2.0 \end{pmatrix}.$$

Because the co-domain has the same dimension as the range, the two are equal, and thus each point in the plane has a pre-image under this mapping.



Regardless of which vector $\mathbf{v} \in \mathbf{R}^2$ you pick, there are infinitely many vectors $\mathbf{u} \in \mathbf{R}^3$ that map under G onto $\mathbf{v}$.

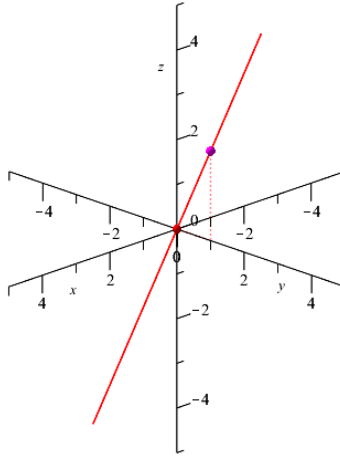
A basis for the null space of G

We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^3$ will be all vectors in $\mathbf{R}^3$ that are mapped by G onto the zero vector $\mathbf{0}_2$.

For the map G , we have $\begin{pmatrix} 1.6 & 0.4 & -0.2 & \vdots & 0 \\ 2.0 & -2.0 & 1.0 & \vdots & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 1 & \vdots & 0 \\ 0 & 2 & -1 & \vdots & 0 \end{pmatrix}$. In this example, because the rank of the matrix is less than the number of columns (that is, the rank is less than the dimension of the domain, or $\text{rank}(A) < \dim(\mathbf{R}^3)$), there are infinitely many vectors that map onto the zero vector $\mathbf{0}_2$. Solving for this basis, we have $\alpha_2 = 0.5\alpha_3, \alpha_1 = 0$. Thus, every vector in the null space is of the form

$$\begin{pmatrix} 0 \\ 0.5\alpha_3 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix}.$$

Thus, for example, $-6.8 \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3.4 \\ -6.8 \end{pmatrix}$ will map to $\mathbf{0}_2$ under this mapping. This subspace is shown in this graph:



Every other point off of this line is mapped to a non-zero vector in $\mathbf{R}^2$.

Finding all pre-images of a given vector

For the map G , given $\mathbf{v} = \begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix}$, we have

$$\left(\begin{array}{ccc|c} 1.6 & 0.4 & -0.2 & 3.4 \\ 2.0 & -2.0 & 1.0 & 3.0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 0 & 2 & -1 & 1 \end{array} \right).$$

From the row-echelon form, we have,

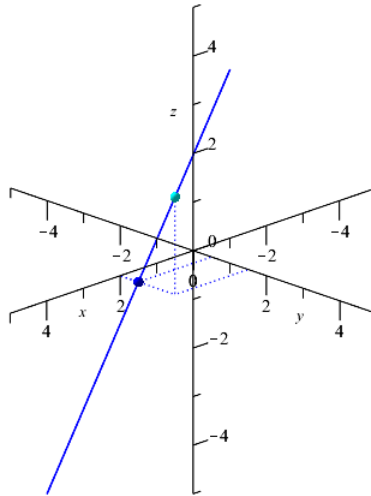
1. the rank of the matrix equals the dimension of the co-domain, so it is always guaranteed that at least one solution exists, and
2. the rank of the matrix is one less than the number of columns, so there is one free variable.

Thus, assigning a free variable to Column 3, as this column does not have a leading non-zero entry, we have that the solutions are $\alpha_2 = 0.5 + 0.5\alpha_3$, $\alpha_1 = 2.0$, so the collection of all solutions is of the form

$$\begin{pmatrix} 2 \\ 0.5 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix}.$$

To visualize this line in $\mathbf{R}^3$, the following graph this line together with the two points $\begin{pmatrix} 2 \\ 0.5 \\ 0 \end{pmatrix}$ and

$$\begin{pmatrix} 2 \\ 0.5 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1.5 \\ 2 \end{pmatrix} :$$

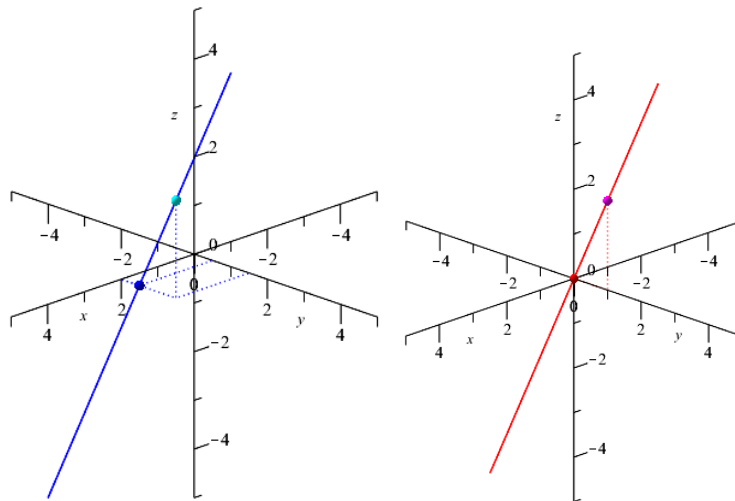


This does not form a subspace in $\mathbf{R}^3$. You will note, however, that

$$\begin{pmatrix} 2 \\ 0.5 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix};$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix}$ to be the image, we see that it is in the set of all solutions.

Also, if you compare this and the null space, you will see that both are parallel in space.



Properties of G

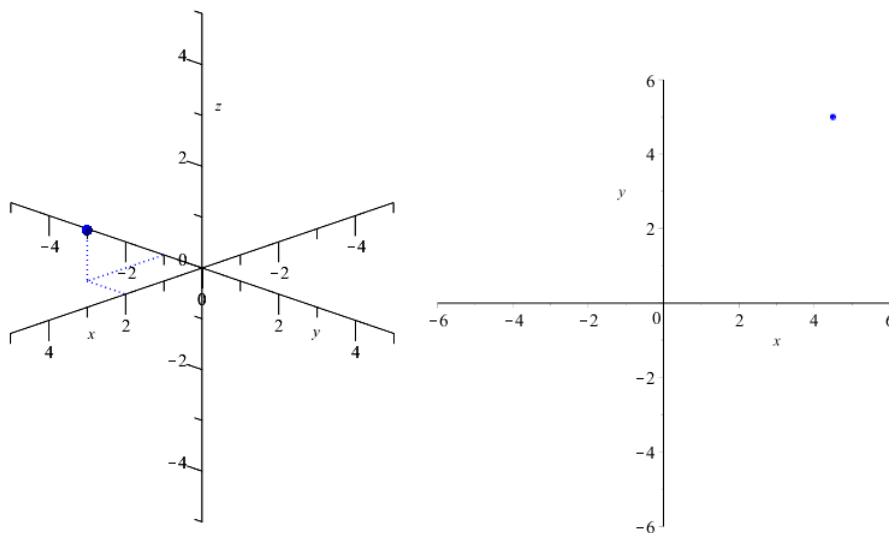
The operator G is onto (every vector in the co-domain $\mathbf{R}^2$ has a pre-image) but not one-to-one (every vector in $\mathbf{R}^2$ that has a pre-image has multiple pre-images).

Example 7: $H: \mathbf{R}^3 \rightarrow \mathbf{R}^2$

Given the linear mapping $H: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ where $H = \begin{pmatrix} 4.5 & 1.8 & -2.7 \\ 5.0 & 2.0 & -3.0 \end{pmatrix}$ is a 2×3 matrix, the dimension of the domain is 3 and the dimension of the co-domain is 2. Because the dimension of the domain is greater than the dimension of the co-domain, the mapping can never be one-to-one—every vector that has a pre-image in the co-domain will have infinitely many pre-images.

Image of a vector under H

If $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \mathbf{R}^3$, $H\mathbf{u} = (-2)\begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix} + (-1)\begin{pmatrix} 1.8 \\ 2.0 \end{pmatrix} + 1\begin{pmatrix} -2.7 \\ 3.0 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix} \in \mathbf{R}^2$, and thus the image of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ under H is $\begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix}$, as shown in the following two graphs:



A pre-image of the vector $\begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix}$ is the vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. As the dimension of the domain is larger than that of the co-domain, there must be infinitely many vectors that map onto $\begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix}$.

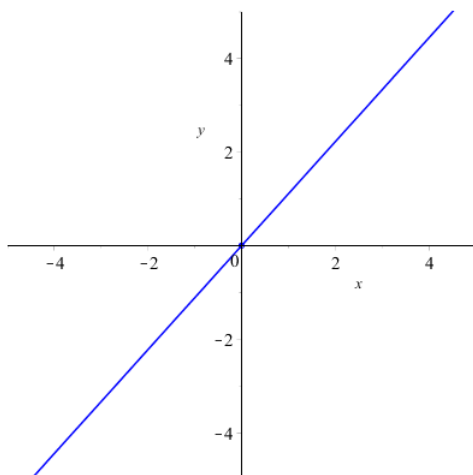
A basis for the range of H

We will now find a basis for the range; that is, all vectors in $\mathbf{R}^2$ that have a pre-image under H will be in the span of this basis. For the map H , we have $\begin{pmatrix} 4.5 & 1.8 & -2.7 \\ 5.0 & 2.0 & -3.0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$. From the row-equivalent matrix in row-echelon form, we have that the rank is 1, so because the dimension of the co-domain is 2, this means that not each vector in the co-domain has a pre-image. Because the rank is less than the number of column vectors, there are infinitely many vectors that map onto each vector in the range. From the row-echelon form, a basis for the range is

Column 1; that is, the column that has a leading non-zero entry in the row-equivalent row-echelon form. Thus, the basis is

$$\begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix}.$$

The range is all scalar multiples of this vector, which forms a subspace that is a line in $\mathbf{R}^2$.



Any vector you pick on this line has infinitely many vectors $\mathbf{u} \in \mathbf{R}^3$ that map under H onto $\mathbf{v}$. Any vector off of this line has no pre-images.

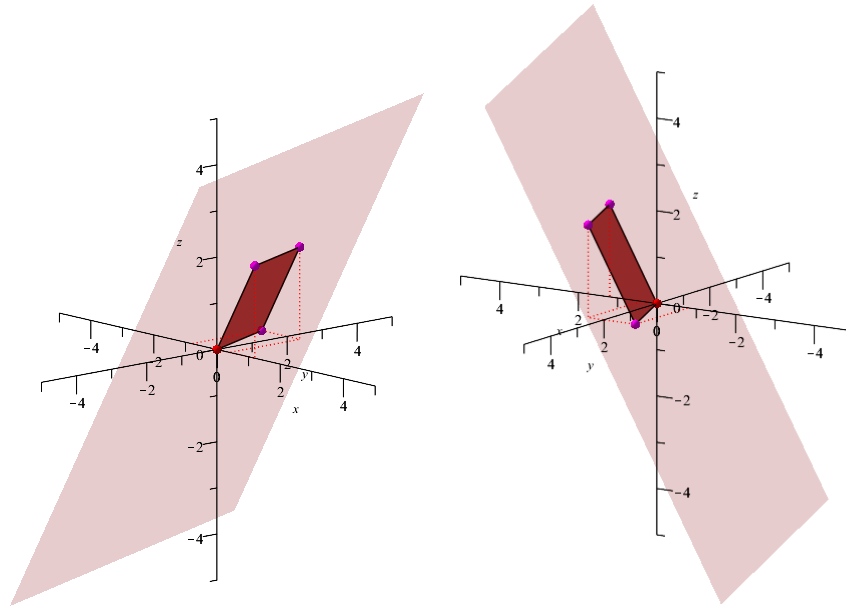
A basis for the null space of H

We will now find a basis for the null space; that is, the span of this basis will be the subspace of $\mathbf{R}^3$ will be all vectors in $\mathbf{R}^3$ that are mapped by H onto the zero vector $\mathbf{0}_2$.

For the map H , we have $\begin{pmatrix} 4.5 & 1.8 & -2.7 & 0 \\ 5.0 & 2.0 & -3.0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. In this example, because the rank of the matrix is less than the number of columns (that is, the rank is less than the dimension of the domain, or $\text{rank}(A) < \dim(\mathbf{R}^3)$), there are infinitely many vectors that map onto the zero vector $\mathbf{0}_2$. We would assign free variables to all columns not having a leading non-zero entry; in this case, Columns 2 and 3. Solving for this basis, we have $\alpha_1 = -0.4\alpha_2 + 0.6\alpha_3$. Thus, every vector in the null space is of the form

$$\begin{pmatrix} -0.4\alpha_2 + 0.6\alpha_3 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \alpha_2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0.6 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, for example, $3.9 \begin{pmatrix} -0.4 \\ 1 \\ 0 \end{pmatrix} - 7.1 \begin{pmatrix} 0.6 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5.82 \\ 3.9 \\ -7.1 \end{pmatrix}$ will map to $\mathbf{0}_2$ under this mapping. This subspace (a plane passing through the origin) is shown in these graphs:



Every other point off of this plane is mapped to a non-zero vector in $\mathbf{R}^2$.

Finding all pre-images of a given vector

For the map H , given $\mathbf{v} = \begin{pmatrix} 4.5 \\ 5.0 \end{pmatrix}$, we have

$$\left(\begin{array}{ccc|c} 4.5 & 1.8 & -2.7 & 4.5 \\ 5.0 & 2.0 & -3.0 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 5 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

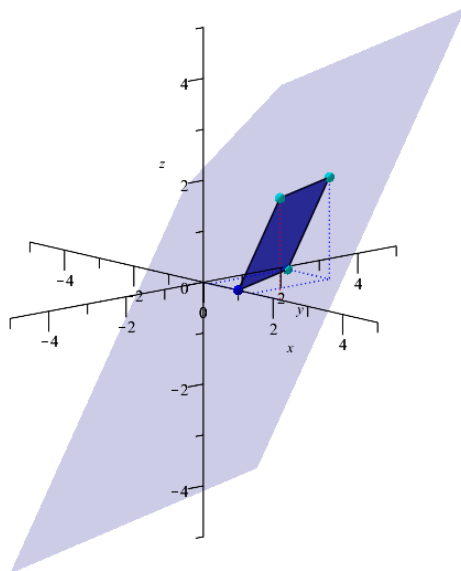
From the row-equivalent matrix in row-echelon form, we have,

1. the rank of the matrix is less than the dimension of the co-domain, so it is not guaranteed a solution exists (although in this case, it does), and
2. the rank of the matrix is two less than the number of columns, so there are two free variables.

Thus, assigning free variables to Columns 2 and 3, as these columns do not have leading non-zero entries, we have that the solutions are $\alpha_1 = 1 - 0.4\alpha_2 + 0.6\alpha_3$, so the collection of all solutions is of the form

$$\begin{pmatrix} 1 - 0.4\alpha_2 + 0.6\alpha_3 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -0.4 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0.6 \\ 0 \\ 1 \end{pmatrix}.$$

To visualize this plane in $\mathbf{R}^3$, the following graph this line together with the two points $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and a few others on the plane:

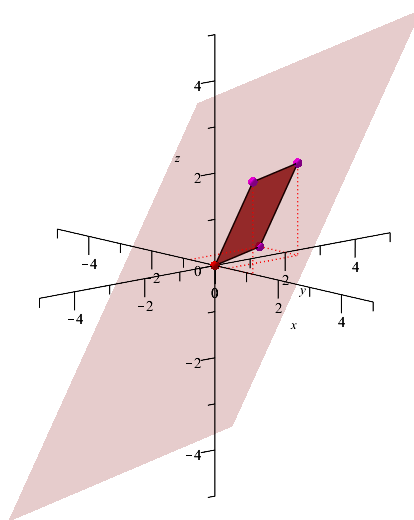
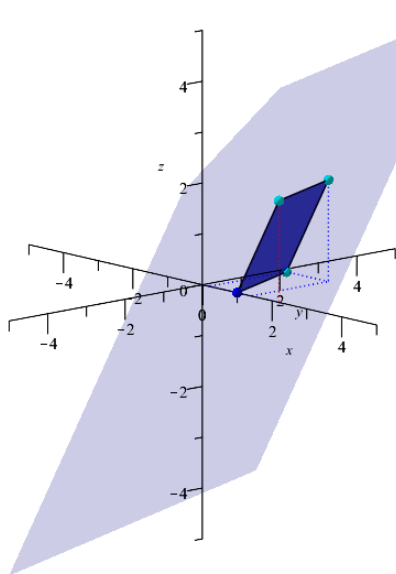


This does not form a subspace in $\mathbf{R}^3$. You will note, however, that

$$\begin{pmatrix} 2 \\ 0.5 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} ;$$

that is, for the one vector for which we found $\mathbf{v} = \begin{pmatrix} 3.4 \\ 3.0 \end{pmatrix}$ to be the image, we see that it is in the set of all solutions.

Also, if you compare this and the null space, you will see that both are parallel planes in $\mathbf{R}^3$.



You will note that, however,

$$\left(\begin{array}{ccc|c} 4.5 & 1.8 & -2.7 & 6.5 \\ 5.0 & 2.0 & -3.0 & 5.0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 5 & 2 & -3 & 5 \\ 0 & 0 & 0 & 2 \end{array}\right),$$

and so no solution exists when the target vector equals $\begin{pmatrix} 6.5 \\ 5.0 \end{pmatrix}$.

Properties of H

The operator H is neither onto (not every vector in the co-domain $\mathbf{R}^2$ has a pre-image) nor one-to-one (every vector in $\mathbf{R}^2$ that has a pre-image has multiple pre-images).