

## Mid-term examination

Douglas Wilhelm Harder [dwharder@uwaterloo.ca](mailto:dwharder@uwaterloo.ca) EIT 4018 x37023

Thursday, October 20, 2022 08:30-10:10 (100 minutes)

Rooms: E7 5353 (140) and E7 5343 (140)

- 1. YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.**
2. The examination is out of 32 marks.
3. No aides.
4. Turn off all electronic media and store them under your desk.
5. There is an eighth page which you can use if there is insufficient room.
6. You may ask only one question during the examination: "May I go to the washroom?"
7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
8. If you think a question is ambiguous, write down your assumptions and continue.
- 9. Do not leave during first hour or after there are only 15 minutes left.**
10. Do not stand up until all exams have been picked up.
11. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
12. The questions are generally in the order of the course material.

1. [2] Prove that  $(1 + x)y = y + yx$  for all scalars  $x$  and  $y$  in a field. Indicate which axiom you are using at each step. Do not apply multiple axioms per step.

2. [1] If we prove that  $-\mathbf{v} = (-1) \cdot \mathbf{v}$  for a vector  $\mathbf{v}$  using only the axioms of a vector space, why are we guaranteed that given a polynomial  $p(x) = x^5 + 5x^3 + x^2 - 4$ , the additive inverse can be found simply by multiplying this polynomial by  $-1$ ? (Very short answer.)

3. [3] Find the rectangular representations of the complex product  $(1 - 2j)(3 + 2j)$  and  $(1 - 2j) \div (3 + 2j)$ .

4. [2] Evaluate the expression  $\frac{(x^2 + 1) + (2x - 2j)}{x - j}$  at the point  $x = 3.2 - 4.3j$  and give the result in the rectangular representation (i.e., in the form  $\alpha + \beta j$ ). You may use the properties of complex numbers to simplify this expression first.

5. [1] What is the relationship between  $z^* z z^*$  and  $z z^* z$  for any complex number  $z$ ?

6. [2] Use the quadratic formula to find the roots of  $x^2 - 1.4x + 0.65$  in the rectangular representation (i.e., in the form  $\alpha + \beta j$ ). If this polynomial was in the denominator of a transfer function, the system is stable if all the roots are in the left-hand complex plane. Is this the case?

7. [1] You are aware that  $2j = \frac{1}{-2.5Cj}$ . What is  $C$ ?

8. [1] Given an angle  $\theta$ , what is the absolute value of the complex number  $z = 3\cos(\theta) + 3\sin(\theta)j$ ?

9. [2] A resistor and a capacitor have an impedance of  $1 - 3j$  and another resistor and an inductor have an impedance of  $2 + 4j$ . If these are put in series (one after the other), the total impedance is the sum of these. What is that sum, and can we replace these two resistors, one capacitor and one inductor with just one resistor and either a capacitor or an inductor? If so, which, and if not, why?

**10. [1]** A polynomial  $p$  of degree less than or equal to 3 has the property that  $p(1) = p(2) = p(7) = p(9) = p(14) = 2$ . What is the polynomial and justify your answer. Hint: consider the polynomial of  $p - 2$ .

**11. [2]** Find the value in the rectangular representation (i.e., in the form  $\alpha + \beta j$ ) of the infinite geometric sum

$$\sum_{k=0}^{\infty} \left( -\frac{4}{5} + \frac{3}{5} j \right)^k .$$

**12. [1]** Normally, a 3-dimensional vector  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  representing the direction a drone is located from the operator where the first component  $x$  is the distance west (negative  $x$  representing distances east), the  $y$  is the distance north, and the  $z$  is the distance up. What stops us, if anything, from having the first component representing the distance speed down, the second representing the distance north-east, and the third representing the distance south-east? (If north is up, west is to the left, and east is to the right.)

**13. [2]** Following up on the previous question, what stops us, if anything, from having the first component representing the distance speed down, the second representing the distance north-east, and the third representing the distance south-west?

**14. [1]** Consider all polynomials of degree exactly equal to two. Is this a subspace of the vector space of all polynomials? Justify your answer.

**15. [2]** Consider all vectors of the form  $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix}$ . Does this form a subspace of vectors in  $\mathbf{R}^3$ ? Justify your answer.

**16. [1]** All real-valued functions of a real variable form a vector space. Are all functions that are periodic with a period  $2\pi$  together with the zero function a subspace of this vector space? Recall a function  $f$  is periodic with period  $2\pi$  if  $f(x + 2\pi) = f(x)$  for all  $x$ .

**17. [2]** Find one vector that is not the zero vector that is perpendicular to the vector  $\mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ .

**18. [2]** In class, we defined the projection of a vector  $\mathbf{u}$  onto a unit vector  $\hat{\mathbf{v}}$  by finding that scalar that minimizes the distance between  $\mathbf{u}$  and  $\alpha\hat{\mathbf{v}}$ , and we saw that the formula is  $\text{proj}_{\hat{\mathbf{v}}}(\mathbf{u}) = \langle \hat{\mathbf{v}}, \mathbf{u} \rangle \hat{\mathbf{v}}$ . Demonstrate that  $\mathbf{u} - \text{proj}_{\hat{\mathbf{v}}}(\mathbf{u})$  is perpendicular to  $\hat{\mathbf{v}}$ .

**19. [3]** Prove or disprove using the axioms of the inner product and properties of a field. If real vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal, then so are vectors  $\alpha\mathbf{u}$  and  $\beta\mathbf{v}$ .

**Axioms of fields:**

1. A field is *closed* under addition and multiplication.
2. Addition and multiplication are *commutative*:  $\alpha + \beta = \beta + \alpha$  and  $\alpha \cdot \beta = \beta \cdot \alpha$ .
3. Addition and multiplication are *associative*:  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$  and  $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ .
4. Multiplication distributes across addition:  $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$ .
5. There is an *additive identity* 0 such that  $\alpha + 0 = \alpha$ .
6. Existence of additive inverses: given  $\alpha$ , there is a  $-\alpha$  such that  $\alpha + (-\alpha) = 0$ .
7. There is a *multiplicative identity* 1 such that  $\alpha \cdot 1 = \alpha$ .
8. Existence of multiplicative inverses for non-zero values: given  $\alpha$ , there is an  $\alpha^{-1}$  such that  $\alpha \cdot (\alpha^{-1}) = 1$ .

**Axioms of vector spaces:****Scalar multiplication axioms**

1. If  $\mathbf{u}$  is a vector, then  $\alpha \mathbf{u}$  is a vector all scalars  $\alpha$ .
2.  $1 \cdot \mathbf{u} = \mathbf{u}$
3.  $\alpha(\beta \mathbf{u}) = (\alpha\beta)\mathbf{u}$

**Vector addition axioms**

4. If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors, then  $\mathbf{u} + \mathbf{v}$  is also a vector.
5.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
6.  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

**The relationships between scalar multiplication and vector addition**

7.  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$
8.  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$

**The zero vector and additive inverses**

9. There exists a  $\mathbf{0}$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
10. Given  $\mathbf{u}$ , there is a  $-\mathbf{u}$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

**Properties of the inner product for real vectors:**

1. *Symmetric* as  $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ ,
2. *Bilinear* as  $\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$  and  $\langle \mathbf{u}, \alpha \mathbf{v} + \beta \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$ , and
3. *Positive definite*,  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$  and  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

**Properties of the inner product for complex vectors:**

1. *Conjugate symmetric*, as  $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$  and
2. *Sesquilinear* as  $\langle \mathbf{u}, \alpha \mathbf{v} + \beta \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$  but  $\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{w} \rangle + \beta^* \langle \mathbf{v}, \mathbf{w} \rangle$ , and
3. *Positive definite*,  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$  and  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

**Properties of a norm:**

1.  $\|\mathbf{u}\| \geq 0$  and  $\|\mathbf{u}\| = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ ,
2.  $\|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|$ , and
3.  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ .

$$\|\mathbf{u}\|_2 \stackrel{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \quad |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \sum_{k=0}^n \alpha r^k = \alpha \frac{1-r^{n+1}}{1-r} \text{ and } \sum_{k=0}^{\infty} \alpha r^k = \frac{\alpha}{1-r}$$

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If you are continuing a question on this page, ensure that at the question, you indicate that you are continuing to answer the question on Page 8.