

NE 112 *Linear algebra for nanotechnology engineers*
FINAL EXAMINATION

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1. **As you sit down, you may immediately rip off the last page (the formula sheet).**
2. **YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.**
3. **The exam will be graded out of 54.**
4. Turn off all electronic media and store them under your desk.
5. If there is insufficient room, use the balance of the last page and indicate you are continuing on that page.
6. You may ask only one question during the examination: “May I go to the washroom?”
7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
8. If you think a question is ambiguous, write down your assumptions and continue.
9. **Do not leave during first hour or after there are only 15 minutes left.**
10. Do not stand up until all exams have been picked up.
11. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.

1. [1] For a given complex number z , suppose you know that $1.2z^3 + 3.2z^2 - 1.5z + 2.3 = 4.53 - 3.25j$. What is this polynomial (the left-hand side of the equation) equal to when evaluated at z^* ?

2. [3] Evaluate the following expression when $z = -224j$, and write the result in the rectangular representation:

$$\frac{z^3 - 2z^2 + 3jz}{-2jz^2 + 4jz + 6}$$

You should make an attempt to simplify the expression first. You will know if you have the right answer.

3. [2] Simplify the expression $-(4\angle 5^\circ)^*(6\angle -7^\circ)^2$ so that the result is of the form $r\angle \theta$. Note that the star is conjugation, and not multiplication.

4. [2] Justify why all two-dimensional vectors of the form $\begin{pmatrix} 0 \\ z \\ z^* \end{pmatrix}$, where z can take on any complex value, is not a subspace of $\mathbb{C}^3$.

5. [2] Given that $x = (2, 1, 1/2, 1/4, 1/8, \dots)$ and $y = (1, 1/4, 1/4^2, \dots)$, find $\text{proj}_y(x)$ given that $\sum_{k=0}^\infty \frac{1}{16^k} = \frac{16}{15}$, $\sum_{k=0}^\infty \frac{1}{8^k} = \frac{8}{7}$, $\sum_{k=0}^\infty \frac{1}{4^k} = \frac{4}{3}$ and $\sum_{k=0}^\infty \frac{1}{2^k} = \frac{2}{1}$.

6. [1] You have a drone that has a camera mounted in the center of the top of the drone. The camera can rotate 360° and can angle down to 5° below the surface defined by the rotors. The center of the drone is at coordinate $\mathbf{u}$ and the unit normal vector that points in the direction perpendicular to the top surface of the rotors is $\hat{\mathbf{n}}$. If an objective is at location $\mathbf{v}$, what calculation would you make to see if the camera can view the objective at $\mathbf{v}$?

6. [1] Apply Gram-Schmidt to the three vectors $\mathbf{u}_1 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$.

7. [2] Write down the system of linear equations that must be solved to find all solutions that balance this reaction:
 $a\text{NaHCO}_3 + b\text{HC}_2\text{H}_3\text{O}_2 = c\text{NaC}_2\text{H}_3\text{O}_2 + d\text{CO}_2 + e\text{H}_2\text{O}$

8. [3] You have four 3-dimensional vectors, and you find that

$$\begin{pmatrix} 1 & 2.4 & 6.8 & 3.2 \\ -1 & -1.4 & -3.8 & 2.3 \\ 5 & 2.0 & 4.0 & 1.0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 & 4 & 1 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Find a basis for the span of these four vectors. What is the dimension of the span? What linear combination of the basis vectors chosen by our algorithm equal those vectors that were discarded by the algorithm?

9. [2] If $A:\mathbf{R}^n \rightarrow \mathbf{R}^m$, for each statement on the right, enter the relevant value of a, b, c or d:

a. $\text{rank}(A) = m$	_____ A is onto
b. $\text{rank}(A) = n$	_____ A is one-to-one
c. $\text{rank}(A) < m$	_____ There are no free variables when solving $A\mathbf{u} = \mathbf{0}_m$
d. $\text{rank}(A) < n$	_____ There is a vector $\mathbf{v}$ in $\mathbf{R}^m$ for which no solution to $A\mathbf{u} = \mathbf{v}$ exists
e. $\text{rank}(A) > m$	
f. $\text{rank}(A) > n$	

10. [3] The determinant is the change in area/volume/hyper-volume for a matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, and $\det(BA) = \det(B)\det(A)$. Explain why the determinant of the identity matrix must be 1. Next, if $\det(R_{\alpha Rk}) = \alpha$, explain why the determinant of a diagonal matrix is the product of the diagonal entries. Finally, if $\det(R_{\alpha Ri \rightarrow Rj}) = 1$, explain why the determinant of an upper-triangular matrix is still equal to the product of the diagonal entries.

11. [2] Assume you have found two complex vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ that are two different solutions to $A\mathbf{u} = \mathbf{v}$ for some complex target vector $\mathbf{v}$. Show that there must be infinitely many solutions to the given system of linear equations by showing that $\alpha\mathbf{u}_1 + (1 - \alpha)\mathbf{u}_2$ is also a solution to $A\mathbf{u} = \mathbf{v}$ for all complex scalars α .

12. [1] What are the eigenvalues of the matrix $A = \begin{pmatrix} 5 & 0 & 2 \\ -2 & -1 & -1 \\ 0 & 5 & 1 \end{pmatrix}$?

13. [5] What is a basis for the null space and a basis for the range of the linear mapping defined by

$$A = \begin{pmatrix} 0.2 & 0.2 & -0.9 & 0.1 & -0.2 \\ 4.0 & 4.0 & 2.0 & 6.0 & 8.0 \\ -0.6 & -0.6 & 4.7 & 0.1 & 1.8 \end{pmatrix}$$

14. [2] Prove that the composition of two isometric operators (each satisfying $\|A\mathbf{u}\|_2 = \|\mathbf{u}\|_2$) is itself isometric. Recall that $(BA)\mathbf{u}$ is by definition $B(A\mathbf{u})$. You must use this fact.

15. [3] Suppose that we sampling data once per second for a total of four samples: (1, 3), (2, 3), (3, 4), (4, 5). What is the least-squares best-fitting line $y = at + b$ that passes through these four points?

16. [4] For linear operators $A, B: \mathbf{C}^n \rightarrow \mathbf{C}^n$, if A and B are normal ($AA^* = A^*A$) and $AB = BA$, show that AB and $A + B$ are normal. You must show each step.

17. [2] Given the PLU decomposition $A = PLU$ where

$$A\mathbf{u} = \begin{pmatrix} 1.2 & 5.9 \\ -1.6 & 1.2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -0.75 & 1 \end{pmatrix} \begin{pmatrix} -1.6 & 1.2 \\ 0 & 6.8 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 13 \\ 0.8 \end{pmatrix},$$

solve this for the vector $\mathbf{u}$ given the target vector $\mathbf{v} = \begin{pmatrix} 13 \\ 0.8 \end{pmatrix}$ using the PLU decomposition shown in class.

18. [1] Recalling that $\text{rank}(A) = \text{rank}(A^T)$, what is the maximum dimension of the range $A: \mathbf{R}^7 \rightarrow \mathbf{R}^4$.

19. [2] If A is a real orthogonal matrix (a real isometric operator), for what values of α is αA still an orthogonal matrix? Indicate which properties you are using to make this statement.

20. [4] Prove that the eigenvalues of a self-adjoint linear operator are real.

21. [4] Given the matrix $A = \begin{pmatrix} 1.25 & 0.75 & 1.25 & -0.25 \\ 0.75 & 1.25 & -0.25 & 1.25 \\ 1.25 & -0.25 & 1.25 & 0.75 \\ -0.25 & 1.25 & 0.75 & 1.25 \end{pmatrix}$, you know the eigenvalues are $-1, 1, 2, 3$. Find an eigenvector corresponding to the eigenvalue 2. Please use partial pivoting.

22. [2] Suppose a plane in $\mathbf{R}^3$ is described by $\mathbf{u} + \alpha \mathbf{v} + \beta \mathbf{w}$ where $\mathbf{u}$ is not the zero vector and the three vectors are linearly independent (so the plane does not pass through the origin). What are the properties of the ideal values of the three vectors that describe this plane, and if these three vectors did not satisfy those ideals, how would we find three corresponding vectors that do have these properties?

Properties of fields:

- 1. A field is *closed* under addition and multiplication.
- 2. Addition and multiplication are *associative*: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
- 3. Addition and multiplication are *commutative*: $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
- 4. There is a 0 such that $\alpha + 0 = \alpha$.
- 5. There is a 1 such that $\alpha \cdot 1 = \alpha$.
- 6. Existence of additive inverses: given α , there is a $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- 7. Existence of multiplicative inverses for non-zero values: given α , there is an α^{-1} such that $\alpha \cdot (\alpha^{-1}) = 1$.
- 8. Multiplication distributes across addition: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Properties of vector spaces: Given any vectors **u**, **v** and **w** and scalars α and β , the vector space is closed under vector addition and scalar multiplication. Additionally,

- 1. $1 \cdot \mathbf{u} = \mathbf{u}$
- 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- 3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- 4. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- 5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
- 6. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
- 7. There exists a **0** such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- 8. Given **u**, there is a $-\mathbf{u}$ such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of the inner product for real vectors:

- 1. *symmetric* as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$,
- 2. *bilinear* as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$, and
- 3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the inner product for complex vectors:

- 1. *conjugate symmetric*, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and
- 2. *sesquilinear* as $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$ but $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^*\langle \mathbf{u}, \mathbf{w} \rangle + \beta^*\langle \mathbf{v}, \mathbf{w} \rangle$, and
- 3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of a norm:

- 1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
- 2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$, and
- 3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$\|\mathbf{u}\|_2 \stackrel{def}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \quad |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r}$

From calculus, recall that for functions of a real variable x mapping on to the reals, e^x and x^2 are both one-to-one but not onto, $x^3 - x$ is onto but not one-to-one, and x and x^3 are both one-to-one and onto, and x^2 and xe^x are neither one-to-one nor onto.

Description	Property	$A: \mathbf{R}^n \rightarrow \mathbf{R}^n$	$A: \mathbf{C}^n \rightarrow \mathbf{C}^n$
Adjoint	$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^*\mathbf{v} \rangle$	The transpose denoted A^T	The conjugate transpose denoted A^*
Self-adjoint	$A = A^*$	A is symmetric	A is conjugate symmetric
Skew-adjoint	$A = -A^*$	A is skew-symmetric	A is conjugate skew-symmetric
Isometric	$\ A\mathbf{u}\ _2 = \ \mathbf{u}\ _2$	A is orthogonal	A is unitary

How to prove by contradiction:

- 1. Assume that the exact opposite of what you want to prove is true.
- 2. Using correct mathematics, show that this assumption leads to a logical contradiction.
- 3. As a truth cannot lead to a contradiction, the assumption must be false.
- 4. Thus, the opposite of the assumption (that is, what you originally wanted to prove) must be true.

Example: prove that $\sqrt{2}$ is irrational (the ancient Greeks figured this one out).

Proof:

Assume $\sqrt{2}$ is rational. If this is true, there must exist two integers m and n such that $\sqrt{2} = \frac{m}{n}$ where m and n share no common factors (that is, $\frac{m}{n}$ is in lowest terms). Thus, $2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$, so $2n^2 = m^2$. Thus, m must be divisible by 2. Therefore, we may write $m = 2\tilde{m}$ where $\tilde{m} = \frac{m}{2}$. Therefore, $2 = \left(\frac{2\tilde{m}}{n}\right)^2 = \frac{(2\tilde{m})^2}{n^2} = \frac{4\tilde{m}^2}{n^2}$, so $2n^2 = 4\tilde{m}^2$ and dividing both sides by 2, we get $n^2 = 2\tilde{m}^2$. However, now we see that n is divisible by 2! Our assumption was that m and n had no common divisors, and yet we have shown that if $\sqrt{2}$ is rational, then both the numerator and denominator must be divisible by 2. Therefore this is a contradiction, and thus $\sqrt{2}$ is not rational; that is, $\sqrt{2}$ is irrational.