

Mid-term examination

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Rooms: E7 5353 (139), RCH 112 (115)

1. **YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.**
2. No aides.
3. Turn off all electronic media and store them under your desk.
4. You may ask only one question during the examination: “May I go to the washroom?”
5. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
6. If you think a question is ambiguous, write down your assumptions and continue.
7. **Do not leave during first hour or after there are only 15 minutes left.**
8. Circle this direction for a bonus mark.
9. Do not stand up until all exams have been picked up.
10. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
11. The questions are in the order of the course material.

1.1 [2] Prove that $0x = 0$ for all scalars x in a field.

1.2 [1] Describe, in your own words, why we worry about defining axioms and then having such obvious theorems such as that shown in Question 1.1 deduced from those axioms.

1.3 [3] Find the rectangular representations of the complex product $(3 - j)(2 + 4j)$ and $(3 + j) \div (2 + 4j)$.

1.4 [4] Prove that $|z| = |z^*|$ for all complex scalars z .

1.5 [2] Give all the properties you can concerning the ratio $\frac{z^*}{z}$ in relation to z .

1.6 [2] Find the value in the rectangular representation (e.g., in the form $\alpha + j\beta$) of the infinite geometric sum

$$\sum_{k=0}^{\infty} \left(\frac{3}{5} - \frac{4}{5}j \right)^k .$$

1.7 [2] Given a polynomial of degree $n = 3$ with real coefficients, describe the possible configurations of the roots describing their characteristics (real, etc.)

2.1 [7] Demonstrate that semi-infinite sequences $x = (x_0, x_1, x_2, \dots)$ with real entries are a vector space by:

1. defining the sum of two such sequences and arguing that their sum must also be in this vector space and give an example,
2. defining the scalar multiple of such sequences and arguing that this scalar multiple must also be in this vector space and give an example,
3. define the zero vector of this vector space, and
4. given such a sequence, define the additive inverse and give an example.

3.1 [3] Consider the vector space of all polynomials $p(x)$. Prove whether or not the following two subsets are subspaces:

1. the set of all polynomials $p(x)$ such that $p(0) = 1$, and
2. the set of all polynomials $p(x)$ such that $p(1) = 0$.

3.2 [2] Consider all points in $\mathbf{R}^3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where $x \geq 0, y \geq 0$ and $z \geq 0$. Demonstrate that this is closed under one vector-space operation but not the other.

3.2 [2] Given two subspaces in a vector space, prove that the intersection of those two subspaces must itself be a subspace. That is, if $\mathbf{u}$ and $\mathbf{v}$ are in the intersection then so must $\alpha\mathbf{u}$ for all scalars α and $\mathbf{u} + \mathbf{v}$.

4.1 [1] Suppose I tell you that the 2-norm of a semi-infinite sequence x is defined as $\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} x_k^2}$ and I tell you that if y is the sequence defined such that $y = (1, 1/4, 1/9, 1/16, \dots)$ and that $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \frac{\pi^2}{6}$, what is $\left\|\frac{1}{6}y\right\|_2$?

4.2 [3] If $x = (-4, -\frac{12}{5}, -\frac{36}{25}, -\frac{108}{125}, \dots)$, what are the 1-, 2- and infinity-norms of this semi-infinite sequence?

5.1 [2] Find two vectors that are not the zero vector and not scalar multiples of each other but are perpendicular to the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 1-j \\ -1+j \end{pmatrix}$. Demonstrate that your vectors are perpendicular by calculating the inner products.

5.2 [4] Recalling that for complex vector spaces, $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k^* v_k$, prove that $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{v} \rangle$ but $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$.

5.3 [2] Find the formula for the projection $\text{proj}_{\mathbf{v}}(\mathbf{u})$ by finding that scalar multiple of $\mathbf{v}$ such that $\mathbf{u} - \alpha \mathbf{v}$ is perpendicular to $\mathbf{v}$.

5.4 [1] For what value of x is there a minimum or maximum of the polynomial $ax^2 + bx + c$ where a is non-zero.

5.5 [2] If x is the infinite sequence $x = (1, 0, \frac{1}{2}, 0, \frac{1}{2^2}, 0, \frac{1}{2^3}, 0, \frac{1}{2^4}, 0, \dots)$ and y is the infinite sequence $y = (0, 1, 0, \frac{1}{3}, 0, \frac{1}{3^2}, 0, \frac{1}{3^3}, 0, \frac{1}{3^4}, 0, \dots)$, given that $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{4}{3}$ and $\sum_{k=0}^{\infty} \left(\frac{1}{9}\right)^k = \frac{9}{8}$, what is $\|x - y\|_2^2$. Recall that the 2-norm of a semi-infinite sequence is the square root of the sum of the squares of the entries.

5.6 [4] Perform the Gram-Schmidt algorithm on the three vectors $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ and

$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix}$. Show your work.

6.1 [4] Describe the three allowable operations we may perform on a system of linear equations.

In secondary school, you may have had the following description of an equation operation:

Add a scalar multiple of any one equation to a scalar multiple of any other equation.

Explain why this is not exactly correct, and how would you correct it.

Properties of fields:

1. A field is *closed* under addition and multiplication.
2. Addition and multiplication are *associative*: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
3. Addition and multiplication are *commutative*: $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
4. There is a 0 such that $\alpha + 0 = \alpha$.
5. There is a 1 such that $\alpha \cdot 1 = \alpha$.
6. Existence of additive inverses: given α , there is a $-\alpha$ such that $\alpha + (-\alpha) = 0$.
7. Existence of multiplicative inverses for non-zero values: given α , there is an α^{-1} such that $\alpha \cdot (\alpha^{-1}) = 1$.
8. Multiplication distributes across addition: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Properties of vector spaces: Given any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and scalars α and β , the vector space is closed under vector addition and scalar multiplication. Additionally,

1. $1 \cdot \mathbf{u} = \mathbf{u}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
6. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
7. There exists a $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
8. Given $\mathbf{u}$, there is a $-\mathbf{u}$ such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of the inner product for real vectors:

1. *symmetric* as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$,
2. *bilinear* as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the inner product for complex vectors:

1. *conjugate symmetric*, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and
2. *sesquilinear* as $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$ but $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{w} \rangle + \beta^* \langle \mathbf{v}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of a norm:

1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$, and
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$$\|\mathbf{u}\|_2 \stackrel{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \quad |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \sum_{k=0}^n \alpha r^k = \alpha \frac{1-r^{n+1}}{1-r} \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha r^k = \frac{\alpha}{1-r}$$