

NE 112 *Linear algebra for nanotechnology engineers*

Sections 001

FINAL EXAMINATION

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Room: DC 1351

1. **YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.**
2. **The exam will be graded out of 47.**
3. Personal notes prepared by the student (hand-written, printed, or typeset).
4. Turn off all electronic media and store them under your desk.
5. If there is insufficient room, use the back of the last page.
6. You may ask only one question during the examination: “May I go to the washroom?”
7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
8. If you think a question is ambiguous, write down your assumptions and continue.
9. **Do not leave during first hour or after there are only 15 minutes left.**
10. Do not stand up until all exams have been picked up.
11. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.

1. [1] A friend tries to convince you that the cross product in $\mathbf{R}^3$ should allow you to claim that all three-dimensional vectors are a field where the cross product is vector multiplication. Does the cross product satisfy the axioms for multiplication in a field? If so, argue this, if not, give a counter-example.

2. [1] With composition of linear operators (a linear map from a vector space into itself), it seems as if many of the properties are shared with fields. What is one axiom of fields that is not shared by collection of all linear operators mapping a vector space into itself?

3. [1] The determinant of a linear operator $\lambda I - A$ where $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a polynomial in λ of degree equal to n . Why is it we are guaranteed that a linear operator has a real eigenvector if n is odd?

4. [2] Prove that if vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\|\mathbf{u} + \mathbf{v}\|_2^2 = \|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2$?

5. [2] Prove that the additive inverse of a vector $\mathbf{u}$ is $(-1)\mathbf{u}$.

6. [2] Suppose you have discrete signals (semi-infinite sequences) $x = (x_0, x_1, x_2, \dots)$ of real values and you consider the subset of these signals of all such signals where only a finite number of entries are non-zero. Does this subset represent a subspace? Justify your answer.

7. [2] Given the vectors

$$\begin{pmatrix} 2.1 \\ 3 \\ -0.9 \end{pmatrix}, \begin{pmatrix} 1.4 \\ 2 \\ -0.6 \end{pmatrix}, \begin{pmatrix} -2.4 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2.4 \end{pmatrix}, \begin{pmatrix} -0.9 \\ 1 \\ 1.7 \end{pmatrix},$$

find an basis for the span. Be sure to show your work.

7. [2] In $\mathbf{R}^5$, what is the matrix that corresponds to the row operation $R_{2 \leftrightarrow 4}$ and what is the matrix that represents the inverse of this operation?

8. [2] Given the line in 3-space represented by $\begin{pmatrix} 202 \\ -97 \\ 99 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, find the appropriate representation of this line $\mathbf{u} + \alpha \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ that minimizes the 2-norm of $\mathbf{u}$.

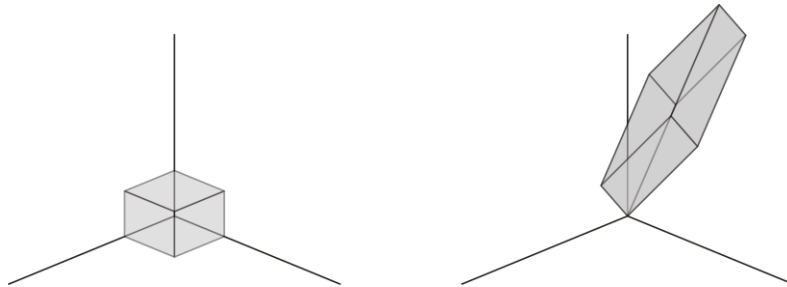
9. [1] Find a non-zero vector that is orthogonal to both $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

10. [1] What is the range of the differentiation operator when it is mapping the vector space of all polynomials of degree less than or equal to 4?

11. [2] Suppose you have found that for a given eigenvalue λ of a linear operator A that the null space of $\lambda I - A$ has a dimension equal to 2, and you find that the two eigenvectors corresponding to λ are $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$. Suppose you apply Gram-Schmidt to these two vectors to get the two orthonormal vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ (you don't have to find these). Are these two orthonormal vectors also eigenvectors of A corresponding to λ ? Justify your answer.

12. [2] If A has an eigenvalue λ with a corresponding eigenvector $\mathbf{v}$, is $\mathbf{v}$ an eigenvector of A^3 ? If not, show this with a counter-example. If it is, find the corresponding eigenvalue.

13. [1] If you multiply every point in the unit cube (all points $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$) by the matrix A , this forms a parallelepiped (a three-dimensional version of a parallelogram). How could you go about finding the volume of this shape?



14. [2] Argue that a matrix representing a linear mapping can only be both one-to-one and onto if the matrix is square.

15. [1] Argue that a linear operator mapping a finite dimensional vector space represented by a square matrix A is invertible if and only if it is both one-to-one and onto. Give an example of a square matrix that is not the zero matrix but is still not invertible (you may choose the dimension).

16. [3] Given the discrete signal (a semi-infinite sequence) $x = (x_0, x_1, x_2 \dots)$, suppose we define the linear operator $Dx = (x_1, x_2 \dots)$. For example, if $x = (1.3, 4.7, 2.9, 8.0, 6.5, \dots)$ then Dx is equal to $(4.7, 2.9, 8.0, 6.5, \dots)$. This is a linear operator mapping an infinite-dimensional vector space into itself; however, it is either one-to-one and not onto, or onto but not one-to-one. Which one is it? Argue that D is not invertible.

17. [1] Eigenvectors corresponding to different eigenvalues are _____
(choose the most appropriate word).

18. [1] Eigenvectors corresponding to different eigenvalues of a symmetric matrix are _____
(choose the most appropriate word).

19. [3] Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

20. [2] Given any linear operator A , show that if A^*A has an eigenvalue, that eigenvalue must be real and greater than or equal to zero.

21. [4] Prove or disprove each of the following:

- If $A: \mathcal{U} \rightarrow \mathcal{U}$ and $B: \mathcal{U} \rightarrow \mathcal{U}$ are isometric (e.g., $A^{-1} = A^*$), then, $A + B$ is isometric.
- If $A: \mathcal{U} \rightarrow \mathcal{U}$ and $B: \mathcal{U} \rightarrow \mathcal{U}$ are isometric, then, BA is isometric.

If you are proving one of these is false with a counter example, you may choose any vector space $\mathcal{U}$ you wish for your counter example.

22. [3] You are given that the PLU decomposition of a matrix A is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 0.8 & 1 & 0 \\ 0 & -0.7 & 1 \end{pmatrix}, U = \begin{pmatrix} 4 & 2.5 & 0 \\ 0 & 5 & -1.5 \\ 0 & 0 & 6 \end{pmatrix}.$$

First, find the matrix A . Second, solve $A\mathbf{u} = \begin{pmatrix} -6.8 \\ 17.6 \\ 1.5 \end{pmatrix}$ for the vector $\mathbf{u}$ appropriately using the PLU decomposition.

23. [2] The matrix $A = \frac{1}{2} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}$ has two eigenvalues, 0 and 1. Calculate $A^{1000} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. You may use any technique or short cuts you wish.

24. [2] Explain how you would find, give a vector $\mathbf{u}$, the closest point in the subspace that is the span of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$.

25. [1] Explain how you would find, give a vector $\mathbf{u}$, the closest point in the subspace that is the span of the vectors $\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_n$ where the vectors form an orthonormal basis for the subspace.

Properties of fields:

1. A field is *closed* under addition and multiplication.
2. Addition and multiplication are *associative*: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
3. Addition and multiplication are *commutative*: $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
4. There is a 0 such that $\alpha + 0 = \alpha$.
5. There is a 1 such that $\alpha \cdot 1 = \alpha$.
6. Existence of additive inverses: given α , there is a $-\alpha$ such that $\alpha + (-\alpha) = 0$.
7. Existence of multiplicative inverses for non-zero values: given α , there is an α^{-1} such that $\alpha \cdot (\alpha^{-1}) = 1$.
8. Multiplication distributes across addition: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Properties of vector spaces: Given any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and scalars α and β , the vector space is closed under vector addition and scalar multiplication. Additionally,

1. $1 \cdot \mathbf{u} = \mathbf{u}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
6. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
7. There exists a $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
8. Given $\mathbf{u}$, there is a $-\mathbf{u}$ such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of the inner product for real vectors:

1. *symmetric* as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$,
2. *bilinear* as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the inner product for complex vectors:

1. *conjugate symmetric*, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and
2. *sesquilinear* as $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$ but $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^*\langle \mathbf{u}, \mathbf{w} \rangle + \beta^*\langle \mathbf{v}, \mathbf{w} \rangle$, and
3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of a norm:

1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$, and
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$$\|\mathbf{u}\|_2 \stackrel{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \quad |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r}$$

From calculus, recall that for functions of a real variable x mapping on to the reals, e^x and x^2 are both one-to-one but not onto, $x^3 - x$ is onto but not one-to-one, and x and x^3 are both one-to-one and onto, and x^2 and xe^x are neither one-to-one nor onto.

Proofs not on the final:

1. That eigenvectors corresponding to different eigenvalues are linearly independent.
2. The Cauchy-Bunyakovskey-Schwarz inequality.
3. That $\text{rank}(A) = \text{rank}(A^*) = \text{rank}(AA^*)$.
4. That the inner product for real vector spaces is linear.

Description	Property	$A: \mathbf{R}^n \rightarrow \mathbf{R}^n$	$A: \mathbf{C}^n \rightarrow \mathbf{C}^n$
Adjoint	$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^* \mathbf{v} \rangle$	The transpose denoted A^T	The conjugate transpose denoted A^*
Self-adjoint	$A = A^*$	A is symmetric	A is conjugate symmetric
Skew-adjoint	$A = -A^*$	A is skew-symmetric	A is conjugate skew-symmetric
Isometric	$\ A\mathbf{u}\ _2 = \ \mathbf{u}\ _2$	A is orthogonal	A is unitary

- How to prove by contradiction:
1. Assume that the exact opposite of what you want to prove is true.
 2. Using correct mathematics, show that this assumption leads to a logical contradiction.
 3. As a truth cannot lead to a contradiction, the assumption must be false.
 4. Thus, the opposite of the assumption (that is, what you originally wanted to prove) must be true.

Example: prove that $\sqrt{2}$ is irrational (the ancient Greeks figured this one out).

Proof:

Assume $\sqrt{2}$ is rational. If this is true, there must exist two integers m and n such that $\sqrt{2} = \frac{m}{n}$ where m and n share no common factors (that is, $\frac{m}{n}$ is in lowest terms). Thus, $2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$, so $2n^2 = m^2$. Thus, m must be divisible by 2. Therefore, we may write $m = 2\tilde{m}$ where $\tilde{m} = \frac{m}{2}$. Therefore, $2 = \left(\frac{2\tilde{m}}{n}\right)^2 = \frac{(2\tilde{m})^2}{n^2} = \frac{4\tilde{m}^2}{n^2}$, so $2n^2 = 4\tilde{m}^2$ and dividing both sides by 2, we get $n^2 = 2\tilde{m}^2$. However, now we see that n is divisible by 2! Our assumption was that m and n had no common divisors, and yet we have shown that if $\sqrt{2}$ is rational, then both the numerator and denominator must be divisible by 2. Therefore this is a contradiction, and thus $\sqrt{2}$ is not rational; that is, $\sqrt{2}$ is irrational.