

NE 112 Linear algebra for nanotechnology engineers
Sections 001
FINAL EXAMINATION
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Rooms: MC 4059,4061

- 1. YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.
- 2. There are 71 marks with a 2 mark bonus.
- 3. No aides.
- 4. Turn off all electronic media and store them under your desk.
- 5. If there is insufficient room, use the back of the previous page.
- 6. You may ask only one question during the examination: “May I go to the washroom?”
- 7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
- 8. If you think a question is ambiguous, write down your assumptions and continue.
- 9. **Do not leave during first hour or after there are only 15 minutes left.**
- 10. Do not stand up until all exams have been picked up.
- 11. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
- 12. Correctly following the instructions in the next statement will yield a bonus of 1.

THIS BLOCK MUST BE COMPLETED USING ALL CAPITAL LETTERS

Last name as appearing in Quest (e.g., HARDER)															
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Legal Given names as appearing in Quest (e.g., DOUGLAS WILHELM)															
D	O	U	G	L	A	S		W	I	L	H	E	L	M	
Common or nick name (entirely optional; e.g., DOUG)															
D	O	U	G												
Student ID Number						2	0	1	2	3	4	5	6		
uWaterloo User ID														@uwaterloo.ca	

I have read the above instructions:

Signature:

Asking any question other than that question noted above.

1. [2] What is the property of fields that allow the following properties of vectors to be true:

- a. Vector addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- b. The existence of a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$

2. [2] Given a polynomial of degree n with real coefficients, describe the number and character of the roots.

3. [2] If A is an orthogonal matrix with $A: \mathbf{R}^5 \rightarrow \mathbf{R}^5$ and $A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4 \ \mathbf{a}_5)$, why must it be true that $\|\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5\|_2^2 = 5$?

4. [2] Using the properties of the inner product and the norm, prove that if $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$.

5. [2] Given the vectors

$$\begin{pmatrix} 2.1 \\ 3 \\ -0.9 \end{pmatrix}, \begin{pmatrix} 1.4 \\ 2 \\ -0.6 \end{pmatrix}, \begin{pmatrix} -2.4 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2.4 \end{pmatrix}, \begin{pmatrix} -0.9 \\ 1 \\ 1.7 \end{pmatrix},$$

find an basis for the span.

6. [3] Apply Gram-Schmidt to the following three vectors:

$$\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 25 \\ 0 \\ 25 \end{pmatrix}, \begin{pmatrix} 10 \\ 2 \\ 5 \end{pmatrix}.$$

7. [2] Suppose you have six four-dimensional vectors and from these you create a matrix A containing these vectors as columns. Suppose also you apply Gaussian elimination with partial pivoting to A and you find that

$$A \sim \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where all of the stars represent non-zero values. Justify why the 2nd vector is not a scalar multiple of the first vector, but that the third vector can be written as a linear combination of the first two vectors.

8. [4] For a linear mapping $A: \mathcal{U} \rightarrow \mathcal{V}$, show that both the null space and range of A are subspaces of $\mathcal{U}$ and $\mathcal{V}$, respectively. It is easiest to show that if $\mathbf{u}_1, \mathbf{u}_2$ are in the null space, so is $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2$, and if $\mathbf{v}_1, \mathbf{v}_2$ are in the range, so is $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$.

9. [2] In $\mathbf{R}^4$, what is the matrix that corresponds to the row operation $R_{4,7;2 \rightarrow 4}$ and what is the matrix that represents the inverse of this operation?

10. [3] Suppose you have performed Gaussian elimination and have found that a particular matrix A is row-equivalent to the following matrix in row-echelon form. Find a basis for the null space of A .

$$A \sim \begin{pmatrix} 2 & 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

11. [2] Instead of having a reflection row operation $R_{2 \leftrightarrow 4}$, you could define a rotation row operation $R_{\theta; 2 \leftrightarrow 4}$ which has the matrix

$$R_{\theta; 2 \leftrightarrow 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

that rotates the 2-4 plane by θ degrees. Argue that the determinant of this matrix is 1 and what would the inverse $R_{\theta; 2 \leftrightarrow 4}^{-1}$ be?

12. [2] Present a valid argument that shows that a square matrix is onto if and only if it is one-to-one.

13. [2] A permutation matrix P is any matrix such that $P\mathbf{u}$ rearranges (or “permutes”) the entries of $\mathbf{u}$. How many different $n \times n$ permutation matrix are there? Present an argument that every permutation matrix is isometric based on the definition of the 2-norm and the property that the entries of a vector are permuted.

14. [3] You are given that the PLU decomposition of a matrix A is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 0.8 & 1 & 0 \\ 0 & -0.7 & 1 \end{pmatrix}, U = \begin{pmatrix} 4 & 2.5 & 0 \\ 0 & 5 & -1.5 \\ 0 & 0 & 6 \end{pmatrix}.$$

First, find the matrix A . Second, solve $A\mathbf{u} = \begin{pmatrix} -6.8 \\ 17.6 \\ 1.5 \end{pmatrix}$ for the vector $\mathbf{u}$ appropriately using the PLU decomposition.

15. [1] Describe what it means if the determinant of a matrix is negative.

16. [2] Based on the physical interpretation of the determinant, if $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, justify why $\det(\alpha A) = \alpha^n \det(A)$. If you wish, you can start with $n = 1$, $n = 2$, $n = 3$ and generalize.

17. [2] Given the matrix $A = \begin{pmatrix} 3.2 & 7 & -1.5 \\ 0 & -3.5 & 7.05 \\ 4 & 2.5 & 0 \end{pmatrix}$, approximate the determinant of $\det(3Id + 0.03A)$

recalling that $\det(Id + \varepsilon A) \approx 1 + \varepsilon \operatorname{tr}(A)$.

18. [2] Give two different examples of classes of matrices for which the inverse can be found easily and in a manner that is guaranteed to be numerically stable. One class should not be a subset of the other.

19. [2] You are given that $A = \begin{pmatrix} 3+j & 1 \\ -1 & 2-j \\ 2 & 3j \\ 2j & 4 \end{pmatrix}$ where $A: \mathbf{C}^2 \rightarrow \mathbf{C}^4$. What matrix $B: \mathbf{C}^4 \rightarrow \mathbf{C}^2$ guarantees that $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, B\mathbf{v} \rangle$ for all vectors $\mathbf{u} \in \mathbf{C}^2$ and all $\mathbf{v} \in \mathbf{C}^4$?

20. [1] If $A: \mathcal{U} \rightarrow \mathcal{V}$ where the two vector spaces are different, explain why A can never be self-adjoint? Alternatively, explain why a self-adjoint linear mapping must be a linear operator.

21. [2] Prove that any linear combination of two symmetric matrices $A, B: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is also symmetric by using the properties of the inner product and the definition of the adjoint (transpose).

22. [2] Given that $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$, show that $(BA)^T = A^T B^T$ by performing both sets of computations.

23. [4] Prove or disprove each of the following:

- If $A: \mathcal{U} \rightarrow \mathcal{U}$, $A - A^*$ is skew-adjoint.
- If $A: \mathcal{U} \rightarrow \mathcal{U}$, $-AA^*$ is skew-adjoint.

If you are proving one of these is false with a counter example, you may choose any vector space $\mathcal{U}$ you wish for your counter example.

24. [2] Prove that if an isometric linear operator has an eigenvalue λ , that eigenvalue must satisfy $|\lambda| = 1$.

25. [2] Argue that if a linear operator has an eigenvalue $\lambda = 0$ that the linear operator is not invertible. There are many ways of answering this question.

26. [5] List five properties of a non-invertible square matrix (that is, a square matrix that is singular).

- 1.
- 2.
- 3.
- 4.
- 5.

27. [3] The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has three eigenvalues 1, 2 and 3. Find a corresponding eigenvector for each.

28. [3] The matrix $A = \frac{1}{2} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}$ has two eigenvalues, 0 and 1. Calculate $A^{1000} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. You may use any technique or short cuts you wish.

29. [2] A matrix A has eigenvalues 2, 3 and 4. What are the eigenvalues of $3Id + 2A - A^2$? Hint: if $\mathbf{u}$ is an eigenvector of A , is $\mathbf{u}$ an eigenvector of $3Id + 2A - A^2$?

30. [3] Given a discrete signal (an infinite sequence) $x = (x_0, x_1, x_2, x_3, x_4, x_5, \dots)$, we define the delay operator as

$$Dx = (x_1, x_2, x_3, x_4, x_5, x_6, \dots),$$

essentially removing the first entry of the signal and moving up all other entries.

- What are all discrete signals that are mapped to the zero signal $(0, 0, 0, 0, 0, 0, \dots)$? That is, what is the null space of D ? Is this operator therefore one-to-one?
- For finite-dimensional linear operators, we saw that being one-to-one is equivalent to being onto. Is this true for infinite-dimensional operators? Justify your answer.
- Consider a geometric sequence where, given a ratio r , we define $g_r = (1, r, r^2, r^3, r^4, r^5, \dots)$. For a given r , is g_r an eigenvector of D ? If so, what is its eigenvalue, and if not, why not?

Bonus. [2] If we define the inner product between two polynomials p and q as $\langle p, q \rangle \stackrel{\text{def}}{=} \int_0^1 p(x)q(x)dx$, apply Gram-Schmidt to the two polynomials 1 and x .

Properties of fields:

- 1. A field is *closed* under addition and multiplication.
- 2. Addition and multiplication are *associative*: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
- 3. Addition and multiplication are *commutative*: $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
- 4. There is a 0 such that $\alpha + 0 = \alpha$.
- 5. There is a 1 such that $\alpha \cdot 1 = \alpha$.
- 6. Existence of additive inverses: given α , there is a $-\alpha$ such that $\alpha + (-\alpha) = 0$.
- 7. Existence of multiplicative inverses for non-zero values: given α , there is an α^{-1} such that $\alpha \cdot (\alpha^{-1}) = 1$.
- 8. Multiplication distributes across addition: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Properties of vector spaces: Given any vectors **u**, **v** and **w** and scalars α and β , the vector space is closed under vector addition and scalar multiplication. Additionally,

- 1. $1 \cdot \mathbf{u} = \mathbf{u}$
- 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- 3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- 4. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- 5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
- 6. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
- 7. There exists a **0** such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- 8. Given **u**, there is a $-\mathbf{u}$ such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of the inner product for real vectors:

- 1. *symmetric* as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$,
- 2. *bilinear* as $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$, and
- 3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of the inner product for complex vectors:

- 1. *conjugate symmetric*, as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$ and
- 2. *sesquilinear* as $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$ but $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{w} \rangle + \beta^* \langle \mathbf{v}, \mathbf{w} \rangle$, and
- 3. *positive definite*, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of a norm:

- 1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
- 2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$, and
- 3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$\|\mathbf{u}\|_2 \stackrel{def}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \quad |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \quad \sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r}$

From calculus, recall that for functions of a real variable x mapping on to the reals, e^x and x^2 are both one-to-one but not onto, $x^3 - x$ is onto but not one-to-one, and x and x^3 are both one-to-one and onto, and xe^x are neither one-to-one nor onto.

Proofs not on the final:

- 1. That eigenvectors corresponding to different eigenvalues are linearly independent.
- 2. The Cauchy-Bunyakovskey-Schwarz inequality.
- 3. That $\text{rank}(A) = \text{rank}(A^*) = \text{rank}(AA^*)$.
- 4. That the inner product for real vector spaces is linear.

Description	Property	$A: \mathbf{R}^n \rightarrow \mathbf{R}^n$	$A: \mathbf{C}^n \rightarrow \mathbf{C}^n$
Adjoint	$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^* \mathbf{v} \rangle$	The transpose denoted A^T	The conjugate transpose denoted A^*
Self-adjoint	$A = A^*$	A is symmetric	A is conjugate symmetric
Skew-adjoint	$A = -A^*$	A is skew-symmetric	A is conjugate skew-symmetric
Isometric	$\ A\mathbf{u}\ _2 = \ \mathbf{u}\ _2$	A is orthogonal	A is unitary

How to prove by contradiction:

1. Assume that the exact opposite of what you want to prove is true.
2. Using correct mathematics, show that this assumption leads to a logical contradiction.
3. As a truth cannot lead to a contradiction, the assumption must be false.
4. Thus, the opposite of the assumption (that is, what you originally wanted to prove) must be true.

Example: prove that $\sqrt{2}$ is irrational (the ancient Greeks figured this one out).

Proof:

Assume $\sqrt{2}$ is rational. If this is true, there must exist two integers m and n such that $\sqrt{2} = \frac{m}{n}$ where m

and n share no common factors (that is, $\frac{m}{n}$ is in lowest terms). Thus, $2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$, so $2n^2 = m^2$. Thus,

m must be divisible by 2. Therefore, we may write $m = 2\tilde{m}$ where $\tilde{m} = \frac{m}{2}$. Therefore,

$2 = \left(\frac{2\tilde{m}}{n}\right)^2 = \frac{(2\tilde{m})^2}{n^2} = \frac{4\tilde{m}^2}{n^2}$, so $2n^2 = 4\tilde{m}^2$ and dividing both sides by 2, we get $n^2 = 2\tilde{m}^2$. However, now

we see that n is divisible by 2! Our assumption was that m and n had no common divisors, and yet we have shown that if $\sqrt{2}$ is rational, then both the numerator and denominator must be divisible by 2. Therefore this is a contradiction, and thus $\sqrt{2}$ is not rational; that is, $\sqrt{2}$ is irrational.