

NE 112 Linear algebra for nanotechnology engineers
Sections 001

MIDTERM EXAMINATION

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2016-10-20T10:40:00P1H30M

Rooms: CPH 3607 < 20661700
RCH 103 20661700-20680000
RCH 212 > 20680000

- 1. YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.
- 2. There are 52 marks.
- 3. No aides.
- 4. Turn off all electronic media and store them under your desk.
- 5. If there is insufficient room, use the back of the previous page.
- 6. You may ask only one question during the examination: “May I go to the washroom?”
- 7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
- 8. If you think a question is ambiguous, write down your assumptions and continue.
- 9. Do not leave during first hour or after there are only 15 minutes left.
- 10. Write your uWaterloo User ID on the last page for a bonus mark.
- 11. Do not stand up until all exams have been picked up.
- 12. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.

Attention:
The questions are in the order of the course material.

THIS BLOCK MUST BE COMPLETED USING ALL CAPITAL LETTERS

Last name as appearing in Quest (e.g., HARDER)																	
H	A	R	D	E	R												
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D	O	U	G	L	A	S		W	I	L	H	E	L	M			
Common or nick name (entirely optional; e.g., DOUG)																	
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I have read the above instructions:

Signature:

Asking any question
other than that
question noted above.

-5

1.1 [2] What is the name of the property that describes that $xy = yx$ in a given field? What is the name of the property that describes that $x(yz) = (xy)z$ in a given field?

$xy = yx$ _____ $x(yz) = (xy)z$ _____

Recall that the French verb *commuter* means “to substitute or switch”.

1.2 [3] Find the rectangular representations of the complex sum $(3 - j) - (2 + j)$ and the complex division $\frac{3-j}{2+j}$.

1.3 [2] Calculate the complex product of the following complex ratios:

$$\frac{1+j}{2-3j} \frac{2-3j}{2-j} \frac{4+j}{3-j} \frac{2-j}{1+j} \frac{-5+j}{4+j} \frac{6-2j}{5-j}.$$

1.4 [4] For two arbitrary complex numbers z and w , circle those statements that are true in general, and cross out those statements that are not true in general.

$$\begin{array}{llllll} wz^* = z^* w & z = (z^*)^* & (wz)^* = w^* z^* & (w-z)^* = w^* - z^* & (z^2)^* = (z^*)^2 \\ \Re(w^*) = \Re(w) & \Im(w^*) = -\Im(w) & \left(\frac{w}{z}\right)^* = \frac{w^*}{z^*} & (w+z)^* = z^* + w^* & \end{array}$$

1.5 [2] Given that $1 - j$ and $-2 + j$ are roots of the polynomial $x^5 + 2x^4 - x^3 - 2x^2 + 10x$, how many roots does this polynomial have, and what are they?

There are _____ roots and they are:

1.6 [2] Find the value in the rectangular representation (e.g., in the form $\alpha + j\beta$) of the infinite geometric sum

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2} - \frac{1}{2}j \right)^k.$$

2.1 [7] Demonstrate that the polynomials with real coefficients of degree less than or equal to three are a vector space by:

1. defining the sum of two such polynomials and arguing that their sum must also be in this vector space and give an example,
2. defining the scalar multiple of such a polynomial and arguing that this scalar must also be in this vector space and give an example,
3. define the zero vector of this vector space and argue that this must be in this vector space, and
4. given such a polynomial, define the additive inverse and give an example.

3.1 [3] In the vector space $\mathbf{R}^3$, describe any subspace of $\mathbf{R}^3$ not equal to the entire space that contains the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$. Show that your subspace you describe is indeed a subspace, and not simply a subset of $\mathbf{R}^3$.

3.2 [2] In the vector space $\mathbf{R}^2$, suppose we consider all vectors of the form $\begin{pmatrix} |\alpha| \\ -|\alpha| \end{pmatrix}$ where α is any real number. Is this set of vectors a subspace and justify your answer.

4.1 [1] Suppose I tell you that the 2-norm of a function f is defined as $\|f\|_2 = \sqrt{\int_{-\infty}^{\infty} f^2(x) dx}$ and I tell you that if g is the function defined such that $g(x) = e^{-\pi x^2}$ then $\int_{-\infty}^{\infty} e^{-2\pi x^2} dx = \frac{1}{2}$, what is $\|3g\|_2$?

4.2 [4] If $\mathbf{u} = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}$, what are the 1-, 2- and infinity-norms of the scalar product $\frac{7}{3}\mathbf{u}$? Find

the normalization of the vector $\frac{7}{3}\mathbf{u}$; that is, find $\frac{\frac{7}{3}\mathbf{u}}{\left\|\frac{7}{3}\mathbf{u}\right\|_2}$.

5.1 [2] If $\mathbf{u} = \begin{pmatrix} 2 \\ j \\ -1 \\ -j \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} j \\ 1 \\ -j \\ 5 \\ 2j \end{pmatrix}$, calculate $\langle (1+j)\mathbf{u}, (1+j)\mathbf{v} \rangle$.

5.2 [4] By going back to the definition of the projection, demonstrate that the projection is *linear*; that is, show that $\text{proj}_{\mathbf{v}}(\alpha\mathbf{u}) = \alpha \text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{v}}(\mathbf{u}_1 + \mathbf{u}_2) = \text{proj}_{\mathbf{v}}(\mathbf{u}_1) + \text{proj}_{\mathbf{v}}(\mathbf{u}_2)$.

5.3 [2] Is the projection $\text{proj}_{\mathbf{v}}(\mathbf{u})$ *linear* in $\mathbf{v}$; that is, is it true that $\text{proj}_{\alpha\mathbf{v}}(\mathbf{u}) = \alpha \text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{v}_1 + \mathbf{v}_2}(\mathbf{u}) = \text{proj}_{\mathbf{v}_1}(\mathbf{u}) + \text{proj}_{\mathbf{v}_2}(\mathbf{u})$? Justify your answer.

5.4 [2] If $\|\mathbf{u}\|_2 = 3$ and $\|\mathbf{v}\|_2 = 4$, what is the maximum possible value of $|\langle 2\mathbf{u}, -3\mathbf{v} \rangle|$?

5.5 [2] If x is the infinite sequence $(1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots)$ and y is the infinite sequence $(1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \dots)$, demonstrate that it is true that $|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$ given that $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{4}{3}$, $\sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k = \frac{6}{5}$ and $\sum_{k=0}^{\infty} \left(\frac{1}{9}\right)^k = \frac{9}{8}$. Recall that the 2-norm of an infinite sequence is the square root of the sum of the squares of the entries, while the inner product is the sum of the products of the entries.

5.6 [4] Perform the Gram-Schmidt algorithm on the three vectors $\mathbf{u}_1 = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} 8 \\ 7 \\ 6 \end{pmatrix}$. Show your work.

6.1 [4] Find the linear combination of the vectors $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ that equals the vector $\mathbf{v} = \begin{pmatrix} -\frac{13}{2} \\ -2 \\ -\frac{5}{2} \end{pmatrix}$. Use the augmented matrix representation and apply row operations to reduce the matrix to row echelon form, after which you may use whatever technique you wish to find the coefficients.