

NE 112 Linear algebra for nanotechnology engineers
Sections 001
FINAL EXAMINATION
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Room: PAC 11,12

- 1. YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.
- 2. There are 70 marks with a 2 mark bonus.
- 3. No aides.
- 4. Turn off all electronic media and store them under your desk.
- 5. If there is insufficient room, use the back of the previous page.
- 6. You may ask only one question during the examination: “May I go to the washroom?”
- 7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
- 8. If you think a question is ambiguous, write down your assumptions and continue.
- 9. **Do not leave during first hour or after there are only 15 minutes left.**
- 10. Do not stand up until all exams have been picked up.
- 11. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.

Attention:
The questions are in the order of the course material.

THIS BLOCK MUST BE COMPLETED USING ALL CAPITAL LETTERS

Last name as appearing in Quest (e.g., HARDER)																			
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Legal Given names as appearing in Quest (e.g., DOUGLAS WILHELM)																			
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Common or nick name (entirely optional; e.g., DOUG)																			
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uWaterloo User ID													@uwaterloo.ca						

I have read the above instructions:

Signature:

Asking any question other than that question noted above.

1 [1] Argue that $r\angle\theta = re^{j\theta}$.

2 [1] Under which conditions is it that $\alpha\beta = \alpha\gamma$ when $\beta \neq \gamma$?

3 [3] Apply Gaussian elimination with partial pivoting to the following matrix to reduce it to row-echelon form. Remember that for partial pivoting, you choose that entry that is largest in absolute value.

$$\begin{pmatrix} 1+2j & 5 & 5j \\ 1-2j & 1+j & 2 \\ 0 & 10-8j & 1 \end{pmatrix}$$

4 [1] The differential operator is linear, and if $\mathbf{P}_5$ is the vector space of all polynomials of degree less than or equal to 5, what is the range of the differential operator $\frac{d}{d\bullet} : \mathbf{P}_5 \rightarrow \mathbf{P}_5$? Here, $\frac{d}{d\bullet}$ represents the derivative of the polynomial with respect to its variable.

5. [3] Find a linearly independent subset of the following collection of six vectors such that the subset continues to have the same span as the original set. You may use any observations to exclude linearly dependent vectors you wish, but justify your reasons.

$$\begin{pmatrix} -1.0 \\ 0.8 \\ 2.0 \\ -1.8 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 1.2 \\ 3.0 \\ -2.7 \end{pmatrix}, \begin{pmatrix} -3.1 \\ 4.4 \\ 5.0 \\ -1.5 \end{pmatrix}, \begin{pmatrix} -1.4 \\ 2.8 \\ 2.0 \\ 3.2 \end{pmatrix}, \begin{pmatrix} -1.8 \\ 5.0 \\ 1.0 \\ 1.1 \end{pmatrix}, \begin{pmatrix} 0.6 \\ 0.0 \\ -2.0 \\ -2.2 \end{pmatrix}.$$

The first pivot is done for you:

$$\begin{pmatrix} 2 & 3 & 5 & 2 & 1 & -2 \\ 0.8 & 1.2 & 4.4 & 2.8 & 5.0 & 0 \\ -1 & -1.5 & -3.1 & -1.4 & -1.8 & 0.6 \\ -1.8 & -2.7 & -1.5 & 3.2 & 1.1 & -2.2 \end{pmatrix} \sim$$

6 [1] Find the quadratic polynomial $y = ax^2 + bx + c$ that passes through all three points
 $(0, 1), (1, -1), (3, 7)$
by writing down the augmented matrix to which Gaussian elimination and backwards substitution must be applied, but do not solve for a, b or c .

7 [2] Find the best-fitting line $y = ax + b$ that passes through the points
 $(0, 3), (1, 2), (2, -1), (3, 0), (4, -1)$
by writing down the augmented matrix to which Gaussian elimination and backwards substitution must be applied, but do not solve for a or b .

8 [2] A pilot is flying her aircraft in the direction $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and a vector to her right is $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$. She will eject in a direction $\mathbf{w}$ perpendicular to the plane containing the vectors $\mathbf{u}$ and $\mathbf{v}$, but it is only safe to eject if the projection of this ejection vector $\mathbf{w}$ onto $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ points in the same direction as $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Is it safe to eject? Justify your answer.



Photograph by Jim Ford, RCAF.

9 [3] Given that $A: \mathbf{R}^3 \rightarrow \mathbf{R}^4$ and $B: \mathbf{R}^4 \rightarrow \mathbf{R}^2$ where these linear mappings are represented by the matrices $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{pmatrix}$, find the composition $C: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ such that $C\mathbf{u} = B(A\mathbf{u})$ for all vectors $\mathbf{u} \in \mathbf{R}^3$.

10 [1] Describe using the concept of matrix multiplication being a linear mapping from one vector space into another, why it makes no sense to try to multiply a 3×4 matrix and a 7×3 matrix, but it does make sense to multiply a 7×3 matrix by a 3×4 matrix.

11 [3] By finding $\langle A\mathbf{u}_1, \mathbf{u}_2 \rangle$, prove that eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ corresponding to different eigenvalues for a self-adjoint linear operator A are orthogonal. You may assume that the eigenvalues of a self-adjoint linear operator are real.

12 [5] Find the eigenvalues and corresponding eigenvectors of the following matrix. Once you have written down the characteristics polynomial (that is, the determinant of $A - \lambda Id$), you may assume that the eigenvalues are -2 , 3 and 8 .

$$\begin{pmatrix} 4.8 & 0 & 2.4 \\ 0 & -2 & 0 \\ 2.4 & 0 & 6.2 \end{pmatrix}$$

13 [5] You have been given that $A = PLU$ where

$$A = \begin{pmatrix} 0 & -0.3 & 4 & 0.1 \\ 0.2 & -3 & 0.2 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 0.6 & 0 & 4.8 \end{pmatrix} = PLU = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 1 & 0 & 0 \\ 0 & 0.1 & 1 & 0 \\ 0 & -0.2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

Solve the system of linear equations defined by

$$A\mathbf{u} = \begin{pmatrix} 8.3 \\ 3.6 \\ 6.0 \\ -0.6 \end{pmatrix}$$

for the unknown vector $\mathbf{u}$ using the P , L and U matrices.

14 [1] In the previous question, one of the matrices is such that, even if you as a student, made a mistake, you would still get the right answer. What is the property of that matrix, and why does it allow you as a student to possibly make an error but still get the right answer?

15 [4] Find bases for both the null space and the range of the linear operator

$$\begin{pmatrix} 2 & 5 & 4 & 0 \\ -0.4 & -0.6 & -0.8 & 0.6 \\ -0.2 & -0.9 & -0.4 & -0.6 \\ -0.4 & 1 & -0.8 & 3 \\ 0 & 0.6 & 0 & 0.9 \end{pmatrix}$$

16 [4] You have been asked to find the eigenvalues and eigenvectors corresponding to the matrix

$$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{7}{4} & -\frac{1}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{7}{4} \end{pmatrix}$$

and you find that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 2$ with corresponding eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 1 \end{pmatrix}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ 0 \end{pmatrix},$$

respectively. However, these are not all mutually orthogonal, and you know that for a symmetric matrix, it is possible to find eigenvectors that are orthonormal. Transform these into three orthonormal vectors.

17 [4] Assuming that A is invertible, prove that AA^* is self-adjoint and that all eigenvalues of AA^* are positive real numbers. Recall that an eigenvalue of a matrix B is positive if $\langle B\mathbf{u}, \mathbf{u} \rangle > 0$ for an eigenvector $\mathbf{u}$.

18 [2] Find the inverse of the matrices

$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 1 & 0 & 0 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0 & 0.3 & 1 \end{pmatrix}.$$

19 [3] For a complex linear operator $A:U \rightarrow U$, first prove that $(\alpha A)^* = \alpha^* A^*$ for a complex scalar α . Next, assuming that A is unitary, what are all possible complex values of α that allow αA to remain unitary? (Note that the star on the matrix is the adjoint, while the star on the complex scalar is the complex conjugate.)

20 [2] For each of the following, either prove the statement or find a counter example:

- a. If A and B are self-adjoint, then $A + B$ is self-adjoint.
- b. If A and B are unitary, then $A + B$ is unitary.

21 [2] Argue that every real matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ when n is odd must have at least one real eigenvalue. Recall that the eigenvalues are the roots of the determinant of $A - \lambda Id$.

22 [2] Given an arbitrary real square matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, show that $\langle \mathbf{u}, A\mathbf{u} \rangle = \left\langle \mathbf{u}, \frac{1}{2}(A + A^T)\mathbf{u} \right\rangle$ for all vectors $\mathbf{u} \in \mathbf{R}^n$.

23 [3] You have found that for a given symmetric matrix, that matrix can be written as $A = EA_EE^*$ where

$$E = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ and } A_E = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Calculate $A^3\mathbf{u}$ where $\mathbf{u} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$

24 [2] If E is the matrix in the previous question, find $\|(2E)(3\mathbf{u})\|_2$ where $\mathbf{u} = \begin{pmatrix} 3 \\ -1 \\ 9 \\ -3 \end{pmatrix}.$

25 [2] If x is the infinite sequence $\left(1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots\right)$ and y is the infinite sequence $\left(1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \dots\right),$ demonstrate that it is true that $|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$ given that $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{4}{3}, \sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k = \frac{6}{5}$ and $\sum_{k=0}^{\infty} \left(\frac{1}{9}\right)^k = \frac{9}{8}.$ Recall that the 2-norm of an infinite sequence is the square root of the sum of the squares of the entries, while the inner product is the sum of the products of the entries.

26 [2] Prove the Pythagorean theorem that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then

$$\|\mathbf{u} + \mathbf{v}\|_2^2 = \|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 .$$

27 [4] Give four properties of matrices $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ that differentiate invertible matrices from non-invertible matrices. You may consider rank; the dimensions of the domain, null space, codomain or range; eigenvalues; determinant; or other properties.

28 [2] Give the definition of the adjoint of a linear operator mapping $A: U \rightarrow V$ and explain (do not prove) why we call the adjoint of a real matrix its *transpose*.

Bonus [2] If the determinant of a matrix is -3 , what is the significance of the magnitude 3 and what is the significance of the negative sign?