ECE604- Stochastic Processes # 2
Fall 2013

Do all problems. Some problems extend the theoretical results we have developed in class. I have indicated the implications of the result.

Problem 1: Let X be a non-negative integer valued r.v. with moment generating function $g_X(z)$.

1. Show that $g_X(z)$ is convex and non-decreasing in $z \in (0, 1]$.
2. If $0 < \mathbb{P}(X_i = 0) < 1$ then $g_X(.)$ is strictly decreasing and if $\mathbb{P}(X_i \leq 1) < 1$ then it is strictly convex.
3. Supose $\mathbb{P}(X_i \leq 1) < 1$ and $\mathbf{E}[X_i] = 1$ then the equation $z = g_X(z)$ has a unique solution in $z \in [0, 1]$ at $z = 1$. If $\mathbf{E}[X] > 1$ then there are two solutions at $z = 1$ and $z = z_0 \in (0, 1)$.

Problem 2: Let $\{X_i\}$ be i.i.d. r.v's. with common distribution $F(x)$.

Show that the distribution of $Y_n = \max\{X_1, X_2, ..., X_n\}$ is given by $F_{max}(x) = F^n(x)$ and the complementary distribution function of $Z_n = \min\{X_1, X_2, ..., X_n\}$ is given by $1 - F_{min}(x) = (1 - F(x))^n$.

Now define $\psi_n = Y_n - Z_n$. The r.v. ψ_n is called the range of $\{X_i\}_{i=1}^n$. Show that the distribution $F_\psi(z) = \mathbb{P}(\psi_n \leq z)$ is given by:

$$F_\psi(z) = n \int_{-\infty}^{\infty} (F(x + z) - F(x))^{n-1} \, dF(x)$$

Problem 3: Let X and Y be independent and identically distributed r.v.'s with $\mathbf{E}[X] < \infty$. Then show that

$$\mathbf{E}[X/X + Y] = \mathbf{E}[Y/X + Y] = \frac{X + Y}{2}$$

Now if $\{X_i\}$ is a sequence if i.i.d. r.v's. using the first part establish that :

$$\mathbf{E}[X_1/S_n] = \frac{S_n}{n}$$

where $S_n = \sum_{i=1}^n X_i$

Problem 4: Let X and Y be independent r.v's with the following distributions:

i) X is $N(0, \sigma^2)$.

ii) Y takes the value 1 with prob. p and -1 with prob q=1-p

Define $Z = X + Y$ and $W = XY$. Then:

i) Find the probability density of Z and its characteristic function.

ii) Find the conditional density of Z given Y.

iii) Show that W is Gaussian. Define $U = W + X$. Show that U is not Gaussian. Find $\mathbf{E}[W]$ and $var(W)$. Are W and X correlated ? Independent ?

This problem shows the importance of the condition of 'jointly Gaussian' in connection with linear combinations of Gaussian r.v's.

Problem 5: Let X and θ be independent r.v's with the following distributions:

$$\begin{aligned} p_X(x) &= 0 \quad x < 0 \\ &= xe^{-\frac{x^2}{2}} \quad x \geq 0 \end{aligned}$$

θ is uniformly distributed in $[0, 2\pi]$.

Define

$$Z_t = X \cos(2\pi t + \theta)$$

Show that for each t, Z_t is Gaussian.

Is the joint distribution of $\{Z_{t_1}, Z_{t_2}\}$ for arbitrary t_1, t_2 Gaussian?

Problem 6: Let X be a Poisson r.v. i.e. X takes integer values $0, 1, 2, ...$ with $\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$; $\lambda > 0$. Compute the mean and variance of X. Find the characteristic function of X. Let $\{X_i\}$ be independent Poisson r.v's. with parameters $\{\lambda_i\}$. Then show that $Y = \sum_{i=1}^{n} X_i$ is a Poisson r.v. with parameter $\sum_i \lambda_i$.

Problem 7 (Advanced problem) (Not to be turned in)

Let X and Y be independent r.v's such that $Z = X + Y$ is Gaussian. Show that X and Y are Gaussian r.v's.

This is a part converse to the statement that sums of jointly Gaussian r.v's are Gaussian. This shows that if a r.v. is Gaussian then we can always find two (or even n) independent Gaussian r.v's such that the given r.v. is the sum of the two r.v's. (or n r.v's). Similarly for the case of the Poisson r.v's . This is related to an important notion of *infinitely divisible* properties of such distributions. In fact it can be shown that if a distribution is infinitely divisible then its characteristic function is a product of a Gaussian characteristic function and a characteristic function of the Poisson type.

Problem 8: (Advanced problem) (Not to be turned in)

a) Show that a random variable is standard normal if and only if

$$\mathbf{E}[f'(X) - Xf(X)] = 0$$

for all continuous and piecewise differentiable functions $f(.) : \Re \to \Re$ with $E[f'(Z)] < \infty$ for $Z \sim N(0, 1)$.

b) Let $h : \Re \to \Re$ be a piecewise continuous function. Show that there exists a function f which satisfies the so-called Stein equation:

$$h(x) - \Phi h = f'(x) - xf(x)$$

where $\Phi h = E_N[h]$ where $E_N[.]$ denotes the expectation w.r.t. the standard normal distribution.

This problem is related to defining a metric in the space of probability distributions. It allows us to "measure" how "different" a given distribution is from a normal distribution.