

EE604- Stochastic Processes # 1
Fall 2013

This problem set should serve as a diagnostic problem set. It covers the background in probability and random variables that is required.

Problem 1: A random variable $X(\omega)$ takes non-negative values and has the probability distribution function given by:

$$\begin{aligned} F(x) = \mathbf{Pr}\{X(\omega) \leq x\} &= 1 - e^{-2x} \quad ; x \geq 0 \\ &= 0 \quad otherwise \end{aligned}$$

- a) Calculate the following: $\mathbf{Pr}\{X(\omega) \leq 1\}$, $\mathbf{Pr}\{X(\omega) > 2\}$ and $\mathbf{Pr}\{X(\omega) = 3\}$.
- b) Find the probability density function $p_X(x)$ of $X(\omega)$.
- c) Let $Y(\omega)$ be a r.v. obtained from $X(\omega)$ as follows:

$$\begin{aligned} Y(\omega) &= 0 \quad if \quad X(\omega) \leq 2 \\ &= 1 \quad if \quad X(\omega) > 2 \end{aligned}$$

Find the probability density function $P_Y(y)$ for $Y(\omega)$.

Problem 2: Let $X(\omega)$ be a real valued r.v. with distribution function $F(x)$. You may assume that $\mathbf{E}|X| < \infty$.

- a) If $X(\omega) \geq 0$, show that:

$$\mathbf{E}[X(\omega)] = \int_0^\infty (1 - F(x))dx$$

- b) If $-\infty < X(\omega) < \infty$ then:

$$\mathbf{E}[X(\omega)] = \int_0^\infty (1 - F(x))dx - \int_{-\infty}^0 F(x)dx$$

Problem 3: Let $X(\omega)$ and $Y(\omega)$ be jointly distributed, non-negative random variables. Show that:

$$\mathbf{P}(X + Y > z) = \mathbf{P}(X > z) + \mathbf{P}(X + Y > z \geq X)$$

and

$$\int_0^\infty \mathbf{P}(X + Y > z \geq X)dz = E[Y]$$

Problem 4: Let $X(\omega)$ and $Y(\omega)$ be independent r.v.'s with probability density functions (p.d.f) given by:

$$p_X(x) = \frac{1}{2} \left[\delta(x+1) + \delta(x-1) \right]$$

$$p_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

where $\delta(\cdot)$ is the Dirac delta function.

Let $Z(\omega) = X(\omega) + Y(\omega)$ and $W(\omega) = X(\omega)Y(\omega)$.

- Find the p.d.f. of Z i.e. $p_Z(z)$.
- Find the conditional densities: $p_{Z/X}(z/x = -1)$ and $p_{Z/X}(z/x = 1)$.
- Find the mean of W i.e. $\mathbf{E}[W(\omega)]$ and $\text{cov}(W) = \mathbf{E}(W(\omega) - \mathbf{E}[W])^2$. Are Y and W uncorrelated? Independent?

Problem 5: Suppose $X_1(\omega)$ and $X_2(\omega)$ be two jointly distributed r.v.'s whose joint density is given by:

$$p(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}}$$

Let $Y(\omega) = \sqrt{X_1^2(\omega) + X_2^2(\omega)}$.

Find $\mathbf{E}[X_1|Y]$.

Problem 6: Let $X_1(\omega)$ and $X_2(\omega)$ be two zero mean r.v.'s.

Suppose we find a linear combination $Y(\omega) = X_1(\omega) + \alpha X_2(\omega)$ which is independent of $X_2(\omega)$.

Find $\mathbf{E}[X_1|X_2]$.

Problem 7: Let $X(\omega)$ be a r.v. with $\mathbf{E}[X^2] < \infty$. Let $m = \mathbf{E}[X]$. Chebychev's inequality states that:

$$\Pr(|X(\omega) - m| \geq \varepsilon) \leq \frac{\text{var}(X)}{\varepsilon^2}$$

Show that the following one-sided version of Chebychev's inequality holds:

$$\Pr(X(\omega) - m \geq \varepsilon) \leq \frac{\text{var}(X)}{\text{var}(X) + \varepsilon^2}$$

where $\text{var}(X)$ is the variance of $X(\omega)$.

The second result is often called Cantelli's inequality.