ECE 604- PSET 5 Solution

1. a) Define $Y_n = X_{N-n}$ for some N fixed. It can be taken to be 0 wlog.

   We need to show $\mathbb{P}\{Y_{n+1} \in A | Y_0, Y_1, \cdots, Y_n\} = \mathbb{P}\{Y_{n+1} \in A | Y_n\}$.

   By abuse of notation:

$$\begin{aligned}
\mathbb{P}(Y_{n+1} \in A | Y_0, \cdots, Y_n) &= \frac{\mathbb{P}(Y_{n+1} \in A, Y_0, Y_1, \cdots, Y_n)}{\mathbb{P}(Y_0, \cdots, Y_n)} \\
&= \frac{\mathbb{P}(X_{-(n+1)} \in A) \prod_{k=n}^{0} \mathbb{P}(X_{-k+1} | X_{-k})}{P(X_{-n} \prod_{k=n}^{0} \mathbb{P}(X_{-k+1} | X_{-k})} \\
&= \frac{\mathbb{P}(X_{-(n+1)} \in A)}{\mathbb{P}(X_{-n})} \\
&= \mathbb{P}(Y_{n+1} \in A | Y_n)
\end{aligned}$$

   Establishing that it is Markov in reverse time.

   b) The reverse chain need not be homogeneous even when $\{X_n\}$ is unless $\{X_n\}$ is stationary.

   To see this: Let $Q_{i,j}(n) = \mathbb{P}(Y_{n+1} = j | Y_n = i)$

   Then:

$$\begin{aligned}
Q_{i,j}(n) &= \mathbb{P}(Y_{n+1} = j | Y_n = i) \\
&= \frac{\mathbb{P}(Y_{n+1} = j, Y_n = i)}{\mathbb{P}(Y_n = i)} \\
&= \frac{P_{j,i} \pi_j(-n)}{\pi_i(-n)}
\end{aligned}$$

   where $\pi_j(-n) = \mathbb{P}(X_{-n} = j)$. Since this depends on n the transition probabilities of $\{Y_n\}$ depend on n and so the process is not a HMC.

   If $\{X_n\}$ is stationary then $\pi_j(n) = \pi_j$ and then:

$$Q_{i,j} = P_{j,i} \frac{\pi_j}{\pi_i}$$

   and then $\{Y_n\}$ is a HMC.

   c)

$$\mathbb{P}(X_{n-1} = i, X_{n+1} = j | X_n = k) = \mathbb{P}(X_{n-1} = i | X_n = k) \mathbb{P}(X_{n+1} = j | X_n = k)$$

   by the conditional independence of the future and past given the present property of Markov processes.

   Therefore using the previous result above we obtain:

$$\mathbb{P}(X_{n-1} = i, X_{n+1} = j | X_n = k) = P_{ik} \frac{\pi_i}{\pi_k} P_{kj}$$

2. This problem has been solved in the notes. See section on coupling. (End of the Chapter)

3. Let $f_{ij} = \mathbb{P}(\tau_j < \infty | X_0 = i) = \mathbb{P}(X_n = j \; eventually | X_0 = i)$

   The first part is easy.

   We now show:

$$\sup_n P_{ij}^{(n)} \leq f_{ij} \leq \sum_{m=1}^{\infty} P_{ij}^{(m)}$$

   From 1-step analysis:

$$P_{ij}^{(n)} = \sum_{m=1}^{n} f_{ij}^{(m)} P_{jj}^{(n-m)}$$

   Hence:

$$\sup_n P_{ij}^{n)} \leq \sum_{m=1}^{\infty} f_{ij}^{(m)} = f_{ij}$$

   Now:

$$N_j = \sum_{n=1}^{\infty} \mathbf{1}_{[X_n=j]}$$

   So:

$$\mathbf{E}_i[N_j] = \sum_{n=1}^{\infty} \mathbf{E}_i[\mathbf{1}_{[X_n=j]}] = \sum_{m=1}^{\infty} P_{ij}^{(m)}$$

   Now:

$$f_{ij} = \sum_{m=1}^{\infty} \mathbb{P}_i(\tau_j = m) = \sum_{m=1}^{\infty} \mathbf{E}[\mathbf{1}_{[\tau_j=m]}]$$

   Noting that:

$$\mathbf{1}_{[\tau_j=m]} = \mathbf{1}_{[X_1 \neq j, X_2 \neq j, X_{m-1} \neq j, X_m = j]} \leq \mathbf{1}_{[X_m=j]}$$

   The result follows since

$$f_{ij} = \sum_{m=1}^{\infty} \mathbf{E}_i[\mathbf{1}_{[\tau_j=m]}] \leq \sum_{m=1}^{\infty} \mathbf{E}_i[\mathbf{1}_{[X_m=j]}] = \sum_{m=1}^{\infty} P_{i,j}^{(m)}$$

4. We know from 1-step analysis that:

$$P_{ij}^{(n)} = \sum_{m=1}^{n} f_{ij}^{(m)} P_{jj}^{(n-m)}$$

   To prove this result we use the following result

   Let $u_0 = 1$ and $\sum_{k=1}^{\infty} f_k = 1$, with $f_0 = 0$ and

$$u_n = \sum_{k=1}^{n} f_k u_{n-k}$$

   Then:

$$\lim_{n\to\infty} u_n = \frac{1}{\sum_{k=1}^{\infty} k f_k}$$

Proof: Take z-transforms on both sides with $U(z) = \sum_{k=0}^{\infty} u_k z^k$ and $F(z) = \sum_{k=0}^{\infty} f_k z^k = \sum_{k=1}^{\infty} f_k z^k$.

Now $\sum_{k=1}^{\infty} u_k z^k = U(z) - 1$ since $u_0 = 1$. Therefore taking z-transforms on both sides

$$U(z) - 1 = F(z)U(z)$$

or

$$U(z) = \frac{1}{1 - F(z)}$$

¿From the final value theorem:

$$\lim_{n\to\infty} u + n = (1 - z)U(z)|_{z=1}$$

Hence:

$$\lim_{n\to\infty} u_n = \frac{1 - z}{1 - F(z)}|_{z=1}$$

.

Noting $F(1) = \sum_{k=0}^{\infty} f_k = 1$ by assumption, we have via l'Hospital's rule:

$$\begin{aligned} \lim_{n\to\infty} u_n &= \frac{-1}{-F'(z)}|_{z=1} = \frac{1}{\sum_{k=1}^{\infty} k f_k z^{k-1}}|_{z=1} \\ &= \frac{1}{\sum_{k=1}^{\infty} k f_k} \end{aligned}$$

Applying this result taking $u_n = P_{jj}^{(n)}$ and $f_n = f_{jj}^{(n)}$ we obtain:

$$\lim_{n\to\infty} P_{jj}^{(n)} = \frac{1}{\sum_{n=1}^{\infty} n f_{jj}^{(n)}} = \frac{1}{\mathbf{E}_j[\tau_j]}$$

So if j is recurrent we know $\mathbf{E}_j[\tau_j] = \infty$ while if j is positive recurrent $\mathbf{E}_j[\tau_j] < \infty$.

Now using the fact that if $x_n \to x$ then $\frac{1}{n}\sum_{k=1}^{n} x_k \to x$ it readily follows that

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P_{ij}^{(n)} &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{n} f_{ij}^{(m)} P_{jj}^{(n-m)} \\ &= \lim_{N\to\infty} \sum_{m=1}^{\infty} f_{ij}^{(m)} \frac{1}{N} \sum_{n=1}^{N} P_{jj}^{(n-m)} \mathbf{1}_{[m\leq n]} \\ &= f_{i,j} \lim_{N\to\infty} \frac{1}{N} \sum_{p=0}^{N} P_{jj}^{(p)} = \frac{f_{ij}}{E_j[\tau_j]} \end{aligned}$$

5. $\{X_n\}$ is ergodic which implies that the states are positive recurrent and hence $\mathbf{E}_j[\tau_j] < \infty$. We know that $\pi_j = \frac{1}{\mathbf{E}_j[\tau_j]}$.

Therefore:

$$\begin{aligned}
\mathbf{E}[\tau] &= \sum_{j\in E} \mathbf{E}[\tau|X_0=j]\mathbb{P}(X_0=j) \\
&= \sum_{j\in E} E_j[\tau_j]\pi_j \\
&= \sum_{j\in E} \frac{1}{\pi_j}\pi_j = |E|
\end{aligned}$$

Hence if $|E|=\infty$ the mean return time is infinite. This does not contradict positive recurrence because what positive recurrence states is that for every state the expected return time to that state is finite. Since there are in infinite number of states, cycling through all of them is a countable number of finite terms which is infinite.

6. First, note that the sequence $\{\tau_n\}$ is a sequence of stopping or Markov times with $\tau_n \to \infty$ as $n \to \infty$ since X_n is positive recurrent.

   Let $Y_n = X_{\tau_n}$ then by ergodicity:

$$\pi_Y(i) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[Y_n=i]} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[X_{\tau_n}=i]}$$

   Also by the ergodic theorem for MC:

$$\pi_i = \lim_{n\to\infty} \frac{1}{\tau_n}\sum_{n=1}^{\tau_n} \mathbf{1}_{[X_n=i]}$$

   Furthermore:

$$\lim_{N\to\infty} \frac{N}{\tau_N} = Fraction\ of\ time\ X_n \in Y = \sum_{i\in Y} \pi$$

   Now the number of times in $[0,N]$ Y_n is in state $i \in Y$ i.e. $\sum_{n=1}^{N} \mathbf{1}_{[Y_n=i, i\in Y]}$ is the same as the number of times in $[0,\tau_n]$ that $X_n = i, \imath \in Y$.

   Hence:

$$\begin{aligned}
\pi_Y(i) &= \lim_{N\to\infty} \frac{\tau_N}{N}.\frac{1}{\tau_N}\sum_{n=1}^{N} \mathbf{1}_{[X_{\tau_n}=i]} \\
&= \lim_{N\to\infty} \frac{\tau_N}{N}.\frac{1}{\tau_N}\sum_{n=1}^{N} \mathbf{1}_{[X_n=i]} \\
&= \frac{\pi_i \mathbf{1}_{[i\in Y]}}{\sum_{j\in Y}\pi_j}
\end{aligned}$$

   Now:

$$P = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Let us first find the stationary distribution π that satisfies $\pi = \pi P$ which gives:

$$\begin{aligned}
\pi_a &= 0.25\pi_a + 0.5\pi_b \\
\pi_b &= 0.75\pi_a + 0.25\pi_b + 0.5\pi_c \\
\pi_c &= 0.25\pi_b + 0.5\pi_c
\end{aligned}$$

Using the fact that $\pi_a + \pi_b + \pi_c = 1$ we can solve for π.
i) If $Y = \{a, b\}$ we obtain using the result above:

$$\begin{aligned}
\pi_Y(a) &= \frac{\pi_a}{\pi_a + \pi_b} = \frac{2}{5} \\
\pi_Y(b) &= \frac{\pi_b}{\pi_a + \pi_b} = \frac{3}{5}
\end{aligned}$$

ii) $Y = \{a, c\}$ we obtain:

$$\begin{aligned}
\pi_Y(a) &= \frac{\pi_a}{\pi_a + \pi_c} = \frac{4}{7} \\
\pi_Y(c) &= \frac{3}{7}
\end{aligned}$$

Let us now obtain this result the long way i.e. by calculating the probability transition matrix for the reduced state space chains i.e. for $Y_n = X_{\tau_n}$.
Case i)

$$Q_{a,b} = \mathbb{P}(Y_{n+1} = b | Y_n = a)$$

Then:

$$\begin{aligned}
Q_{ab} &= P_{ab} + P_{ac}P_{cb} + P_{ac}P_{cc}^2 P_{cb} + P_{ac}P_{cc}^3 P_{cb} + \dots \\
&= P_{ab} + P_{ac}\left(\sum_{i=0}^{\infty} P_{cc}^i\right) P_{cb}
\end{aligned}$$

$$= P_{ab} = 0.75 \quad since\ P_{ac} = 0$$

Now $Q_{aa} = 1 - Q_{ab} = 0.25$ and similarly we have

$$Q_{ba} = P_{ba} + P_{bc}\frac{1}{1 - P_{cc}}P_{ca} = P_{ba} = 0.5$$

and $Q_{bb} = 1 - Q_{ba} = 0.5$.
Now solving for $\pi_Y = \pi_Y Q$ we obtain $\pi_Y(a) = \frac{2}{5}$ and $\pi_Y(b) = \frac{3}{5}$ that coincides with our earlier answer.
Case ii) when $Y = \{a, c\}$.

$$\begin{aligned}
Q_{ac} &= P_{ac} + P_{ab}\left(\sum_{i=0}^{\infty} P_{bb}^i\right) P_{bc} \\
&= \frac{3}{4}\frac{1}{1 - \frac{1}{4}}\frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

Therefore $Q_{aa} = \frac{3}{4}$.

Similarly:

$$\begin{aligned} Q_{ca} &= P_{ca} + P_{cb}\left(\sum_{i=0}^{\infty} P_{bb}\right) P_{ba} \\ &= \frac{1}{3} \end{aligned}$$

and hence $Q_{cc} = \frac{2}{3}$.

Once again solving for $\pi_Y = \pi_y Q$ gives: $\pi_Y(a) = \frac{4}{7}$ and $\pi_Y(c) = \frac{3}{7}$.

7. This equation represents the evolution of a $M/G/1$ queue viewed at departure times (to understand what this means you will need to take ECE 605)

   From the recursion:

   $$X_{n+1} = (X_n - 1)^+ + \eta_n$$

   where η_n is an i.i.d sequence independent of $\{X_u, u \leq n\}$ implies that: $\{X_n\}$ is a Markov chain defined on $\{0, 1, 2, \cdots\}$.

   One way to obtain the stability conditions (conditions for positive recurrence) is by using Pakes' lemma which gives:

   $$\begin{aligned} \mathbf{E}[X_{n+1} - X_n | X_n = i] &= -1 + \mathbf{E}[\eta_n], \quad i \geq 1 \\ &= \mathbf{E}[\eta_n] \quad, i = 0 \end{aligned}$$

   Hence if $\mathbf{E}[\eta_n] < 1$ by Pakes' lemma $\{X_n\}$ is positive recurrent.

   For $\mathbf{E}[\eta_n] < 1$ we need $p_0 > 0$, $p_0 + p_1 < 1$ and $\sum_{k=1}^{\infty} k p_k < 1$ where $p_k = \mathbb{P}(\eta_0 = k)$. On the other hand if $\mathbf{E}[\eta_0] \geq 1$ then $\mathbf{E}[X_n] \to \infty$ and the process is not positive recurrent.

8. In this problem we will compute the distribution of the coupling time exactly.

   Let $\{X_n\}$ and $\{Y_n\}$ be two independent Markov chains on $\{1, 2\}$ with the same transition probability matrix given by:

   $$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

   Let the coupling time denoted by τ be defined by:

   $$\tau = \inf\{n \geq 1 : X_n = Y_n | X_0 = 1, Y_0 = 2\}$$

   Then:

   $$\mathbb{P}(\tau > n) = \mathbb{P}(X_1 \neq Y_1, X_2 \neq Y_2, \ldots, X_n \neq Y_n | X_0 = 1, Y_0 = 2)$$

   Now since the chains are independent:

   $$\mathbb{P}(X_0 = X_1, X_2 = X_1, \cdots, X_n = X_{n-1}; Y_0 = Y_1, \ldots, Y_n = Y_{n-1}) = (1-\alpha)^n (1-\beta)^n$$

Clearly since we start out fro different states $X_n \neq Y_n$ if and only if the both jump at the same time i.e. X from 1 to 2 and Y from 2 to 1.

Probability of 1 simultaneous jump is $\binom{n}{1}\alpha(1-\alpha)^{n-1}\beta(1-\beta)^{n-1}$

Similarly they will be unequal iff they continue to have simultaneous jumps. For 2 simultaneous jumps the probability is $\binom{n}{2}\alpha^2(1-\alpha)^{n-2}\beta^2(1-\beta)^{n-2}$, etc.

Hence:

$$
\begin{aligned}
\mathbb{P}(\tau > n) &= \sum_{k=0}^{n} \binom{n}{k} (1-\alpha)^{n-k}\alpha^k(1-\beta)^{n-k}\beta^k \\
&= ((1-\alpha)(1-\beta)+\alpha\beta)^n
\end{aligned}
$$

But $(1-\alpha)(1-\beta)+\alpha\beta < 1-(\alpha-\beta)^2 < 1$ since both $\alpha, \beta < 1$.

This establishes the fact that the coupling time has a geometric distribution establishing geometric ergodicity.

9. The solution to this problem is exactly as in Problem 6.

   Indeed let us do it directly:

   BY definition of τ_N we have

$$\sum_{k=1}^{\tau_N} \mathbf{1}_{[X_k \in A]} = N$$

   Therefore:

$$
\begin{aligned}
\lim_{N\to\infty} \frac{N}{\tau_N} &= \lim_{N\to\infty} \frac{1}{\tau_N} \sum_{k=1}^{\tau_N} \mathbf{1}_{[X_k \in A]} \\
&= \mathbf{E}_\pi[\mathbf{1}_{[X_0 \in A]}] \quad \textit{by SLLN for Markov chains}
\end{aligned}
$$

   Therefore:

$$\lim_{N\to\infty} \frac{\tau_N}{N} = \frac{1}{\sum_{i\in A} \pi_i}$$