

ECE604- Stochastic Processes
Problem set # 6– Fall 2013

This is the final problem set. These problems deal with various aspects of Markov chains

1. Let $\{X_n\}$ be an irreducible, homogeneous Markov chain with transition probabilities $p_{i,j}$; $i, j \in E \times E$.
 - a) Show that $\{X_n\}$ is Markov in reverse time.
 - b) Find $P(X_{n-1} = i / X_n = j)$. Is the reverse chain homogeneous?
 - c) Find the probability $P(X_{n-1} = i, X_{n+1} = j / X_n = k)$ assuming that the original process is stationary.
2. Let $\{X_n\}$ be an irreducible, homogeneous MC with finite state space E of cardinality r . Show that if $\min_i j p_{i,j} = \varepsilon > 0$ then: if $\pi^{(n)}$ and $\tilde{\pi}^{(n)}$ denote the distributions of the chain at time n starting from two different distributions then:

$$\sum_{j=1}^r |\tilde{\pi}_j^{(n)} - \pi_j^{(n)}| \leq 2(1 - \varepsilon)^n$$

3. Let $\{X_n\}$ be an irreducible, homogeneous Markov chain defined on (E, P) . Show that for all $i, j \in E$, $f_{i,j} \geq f_{i,k} f_{k,j}$ and

$$\sup_n P_{i,j}^{(n)} \leq f_{i,j} \leq \sum_{n=1}^{\infty} P_{i,j}^{(n)}$$

4. Show that for every Markov chain with countably many states, the limit of $P_{i,j}^{(n)}$ always exists in a Cesaro sense:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{i,j}^{(k)} = \frac{f_{i,j}}{\mu_j}$$

where $\mu_j = E_j[T_j]$ where T_j is the first return time to state j .

5. Let $\{X_n\}$ be a stationary M.C. which is ergodic.

Define:

$$\tau = \inf_n \{n \geq 1 : X_n = X_0\}$$

Show that:

$$E[\tau] = |E|$$

where $|E|$ denotes the cardinality of E . If E is countable, then it follows that $E[\tau] = \infty$. Does this contradict positive recurrence of the states?

6. Let $\{X_n\}$ be an ergodic MC. Assume π , the stationary distribution, exists. Let $Y \subset E$ and define:

$$\begin{aligned}\tau_0 &= \inf_n \{n \geq 1 : X_n \in Y\} \\ \tau_n &= \inf_m \{m > \tau_{n-1} : X_m \in Y\}\end{aligned}$$

Show that the MC $Y_n = X_{\tau_n}$ is stationary and its stationary distribution is given by:

$$\pi_Y(i) = \frac{\pi_i}{\sum_{i \in Y} \pi_i} ; i \in Y$$

Consider the MC with states $\{a, b, c\}$ and transition probability P given by:

$$P = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Let $Y = \{a, b\}$. Find the new transition probability for the “watched” chain. and find $\pi_Y(\cdot)$ and check your answer with the formula above. Repeat the problem with $Y = \{a, c\}$.

7.

8. Let $\{X_k\}$ be a Markov chain defined by:

$$X_{n+1} = (X_n - 1)^+ + \eta_{n+1}$$

where $\{\eta_n\}$ is a sequence of i.i.d r.v's with $P(\eta_k = j) = p_j$. Show that the necessary and sufficient conditions for the Markov chain to be positive recurrent is that: $p_0 > 0$, $p_0 + p_1 < 1$ and $\sum_k k p_k < 1$

9. Let X_n and Y_n be two independent MC's on $E = \{1, 2\}$ with:

$$P = \{p_{i,j}\}$$

and: $p_{1,1} = 1 - \alpha$ and $p_{2,2} = 1 - \beta$ for $\alpha, \beta \in (0, 1)$. Let τ be the first time when $X_n = Y_n$. Compute the probability distribution $P(\tau > n)$ when $X_0 = 1$ and $Y_0 = 2$.

10. Let $\{X_n\}$ be an irreducible positive recurrent homogeneous Markov chain with stationary distribution π . Let A be a subset of E . and $\{\tau_k\}$ be the seq of return times to A . Show that:

$$\lim_{k \rightarrow \infty} \frac{\tau_k}{k} = \frac{1}{\sum_A \pi_i}$$