EE604- Stochastic Processes Problem Set # 5
Fall 2013

These are some problems on SLLN, and second-order processes.

1. a) Show $\sqrt{\lambda}W_{\frac{t}{\lambda}}$ is a standard Brownian motion process.

   b) Let $\{X_t;\ -\infty < t < \infty\}$ be a 0 mean Gaussian process with $\mathbf{E}[X_t X_s] = e^{-\lambda|t-s|}$. Express X_t for $t \geq 0$ in the form
   $$X_t = f(t) W_{\frac{g(t)}{f(t)}}$$
   where $\{W_t; t \geq 0\}$ is a standard Brownian motion.

2. Let $\{X_n\}$ be a sequence of non-negative identically distributed r.v's with $\mathbf{E}[X_n] < \infty$. Then show that :
   $$\frac{X_n}{n} \stackrel{a.s}{\to} 0$$

3. Let $\{X_n\}$ be a second order sequence (mean 0). Show that a necessary and sufficient condition for $\{X_n\}$ to converge in the mean square is that:
   $$\mathbf{E}[X_n X_m] \to C$$
   as $n, m\ \to \infty$ independently and C is a constant.

   Using this result show the following:

   Let $\{X_n\}$ be the discrete-time Gauss Markov process defined by:
   $$X_{n+1} = aX_n + bw_n$$
   where $X_0 \sim N(0, \sigma^2)$ and $\{w_n\}$ is an i.i.d. $N(0,1)$ sequence independent of X_0. Define $R_k = E[X_k^2]$.

   If $a > 1$ show that $\frac{X_n}{a^n}$ converges in the mean square.

4. The interval $[0,1]$ is partitioned into sub-intervals of length $p_1, p_2, \ldots, p_n$ with $\sum_{i=1}^{n} p_i = 1$. The entropy of this partition is defined as:
   $$h = -\sum_{i=1}^{n} p_i \log p_i$$
   Let $\{X_i\}$ be i.i.d $U[0,1]$ (uniform) random variables. Let $Z_m(i)$ denote the number of $X_1, \ldots, X_m$ which lie in the ith. interval of the partition. Show that:
   $$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}$$
   satisfies:
   $$\frac{\log R_m}{m} \to -h\ a.s\ \ as\ m \to \infty$$
   .

5. Test whether each of these functions can be the covariance function of some w.s.s. process.

   (a) $R(t,s) = 1 - |t-s|,\ 0 \le |t-s| \le 1$ and $R(t,s) = 0$ otherwise.

   (b) $R(t,s) = e^{-a|t-s|},\ -\infty < t, s < \infty$

   (c) $R(t,s) =!,\ 0 \le |t-s| \le 1$ and $R(t,s) = 0$ otherwise.

6. Let $R(t)$ be the covariance of a w.s.s. process and $\int_{-\infty}^{\infty} |R(t)|dt < \infty$.

   Define:

   $$R_T(t) = R(t)\mathbf{1}_{[|t| \le T}$$

   i.e. we truncate the covariance to a finite interval. Show that $R_T(t)$ need not define a covariance function.

   Hint: Consider $R_T(t) = 1, |t| \le T$ and $R_T(t) = 1 - \frac{|t|}{T}$

   What this means that correlations cannot vanish abruptly.

7. Show that the following function cannot be the covariance function of any discrete-time process:

   $$\begin{aligned} R(n) &= \pi \ n = 0 \\ &= 2 \ n = \pm 5 \\ &= 3 \ n = \pm 15 \\ &= 0 \ for \ all \ other \ n \end{aligned}$$

8. Show that if $R(t,s),\ -\infty < t, s < \infty$ is a covariance function so is $aR(at, as)$ for any $a > 0$.

9. Let $\{W_t\}$ be a standard Brownian motion. Let:

   $$X(t) = \sin(2\pi f t + W_t),\ t \ge 0$$

   Calculate the mean and covariance of $X(t)$.

   Show that:

   $$\begin{aligned} \lim_{t\to\infty} E[X(t)] &= 0 \\ \lim_{T\to\infty} R(T, T+t) &= = \frac{\cos(2\pi f t)}{2} e^{-\frac{|t|}{2}} \end{aligned}$$

   or the process is asymptotically w.s.s.

10. A stochastic process is called lognormal if it is of the form:

    $$Y(t) = e^{X(t)},\ -\infty < t < \infty$$

    where $X(t)$ is a Gaussian process with mean m and covariance $R(\tau) = cov[X(t)X(t+\tau)]$.

    Find the mean and covariance of $y(t)$.