ECE 604- PSET 3 Solution

1. First note that $\mathbf{E}[\frac{S_n}{S_n}] = 1 = n\sum_{i=1}^{n}\mathbf{E}[\frac{X_i}{S_n}]$ which $\rightarrow$ that $\mathbf{E}[\frac{X_i}{S_n}] = \frac{1}{n}$.

   Therefore
$$\mathbf{E}[\frac{S_m}{S_n}] = \sum_{i=1}^{m}\mathbf{E}[\frac{X_i}{S_m}] = m\mathbf{E}[\frac{X_i}{S_n}] = \frac{m}{n}$$

   Also (from the previous problem set) $\mathbf{E}[X_i|S_n] = \frac{1}{n}S_n$ and therefore $\mathbf{E}[S_m|S_n] = \frac{m}{n}S_n$.

   Now: $\mathbf{E}[S_n|S_m] = S_m + \sum_{j=m+1}^{n}\mathbf{E}[X_j|S_m] = S_m + \sum_{j=m+1}^{n}\mathbf{E}[X_j] = S_m + (n-m)$ from the independence of $\{X_{m+1},\ldots,X_n\}$ and S_m.

   In this problem note the difference between the division of two r.v's and conditioning.

2. a)
$$p_{X,Y}(x,y) = \frac{1}{x}, \quad 0 \le y \le x \le 1$$

   Note $P_{X,Y}(x,y)$ can be written as $P_{X,Y}(x,y) = \mathbf{1}_{[y\le x]}\frac{1}{x}, \quad 0 \le x \le 1$.

   Now:
$$p_X(x) = \int_{R_Y} p_{X,Y}(x,y)dy = \int_0^x \frac{1}{x}dy = 1$$

   or X is uniformly distributed in $[0,1]$.

   b) This just follows from the fact that:
$$\begin{aligned} p_{Y|X}(x|y) &= \frac{P_{X,Y}(x,y)}{P_X(x)} = \frac{1}{x} \quad y \in [0,x] \\ &= 0 \;\; \mathit{otherwise} \end{aligned}$$

   or Y is uniformly distributed on $[0,x]$.

   c)
$$\mathbf{E}[Y|X=x] = \int_{R_Y} y p_{Y|X}(y|x)dy = \int_0^x y\frac{1}{x}dy = \frac{x}{2}$$

   d) First note that since $Y \ge 0$ we have:
$$\mathbb{P}(X^2+Y^2<1|X+x) = \mathbb{P}(Y \le \sqrt{1-X^2}|X=x) = \mathbb{P}(Y \le \sqrt{1-x^2})$$

   Therefore if $\sqrt{1-x^2} > x$ or $0 \le x \le \frac{1}{\sqrt{2}}$ we have $P_{Y|X}(Y^2 \le \sqrt{1-x^2}) = 1$ On the other hand if $x > \frac{1}{\sqrt{2}}$ then we know form the fact that Y is uniformly distributed in $[0,x]$ that: $\mathbb{P}_{Y|X}(Y \le \sqrt{1-x^2}) = \frac{\sqrt{1-x^2}}{x}$.

Hence,

$$\begin{aligned}
\mathbb{P}(X^2+Y^2\le 1) &= \int_{R_X} P(X^2+Y^2\le 1|X=x)p_X(x)dx \\
&= \int_0^1 P(X^2+Y^2\le 1|X=x)dx \\
&= \int_0^{\frac{1}{\sqrt{2}}} 1dx + \int_{\frac{1}{\sqrt{2}}}^1 \frac{\sqrt{1-x^2}}{x}dx \\
&= \frac{1}{\sqrt{2}} + \ln\left(\frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) - \frac{1}{\sqrt{2}} \\
&= \ln(1+\sqrt{2})
\end{aligned}$$

3. $Z = X+Y$ and $p_{X,Y}(x,y) = \frac{x+y}{2}e^{-(x+y)}, \quad x,y\ge 0.$
   Using the formula given on Page 15 Chapter 1 of the notes:

$$\begin{aligned}
p_Z(z) &= \int_R p_{X,Y}(v,z-v)dv \\
&= \int_0^z \frac{z}{2}e^{-z}dz \\
&= \frac{z^2}{2}e^{-z}
\end{aligned}$$

   where we use the fact that $P_{X,Y}(x,y)=0$ if $y<0$ or $x<0$.

4. a)

$$\begin{aligned}
\mathbb{P}(U=X) &= \mathbb{P}(X\le Y) = \int_0^\infty P(X\le Y|X=x)P_X(x)dx \\
&= \int_0^\infty e^{-\mu x}\lambda e^{-\lambda x}dx \\
&= \frac{\lambda}{\lambda+\mu}
\end{aligned}$$

   b) Now for $w>0$

$$\mathbb{P}(U\le u, W>w) = \mathbb{P}(U\le u, W>w, X\le Y) + \mathbb{P}(U\le u, W>w, X>Y)$$

   and

$$\begin{aligned}
\mathbb{P}(U\le u, W>w, X\le Y) &= \mathbb{P}(X\le u, Y>X+w) = \int_0^u \lambda e^{-\lambda x}e^{-\mu(x+w)}dx \\
&= \frac{\lambda}{\lambda+\mu}e^{-\mu w}(1-e^{-(\lambda+\mu)u})
\end{aligned}$$

   Similarly,

$$\mathbb{P}(U\le u, W>w, X>Y) = \frac{\mu}{\lambda+\mu}e^{-\lambda w}(1-e^{-(\lambda+\mu)u})$$

   Therefore for $0\le u\le u+w<\infty$ we have:

$$\mathbb{P}(U\le u, W>w) = (1-e^{-(\lambda+\mu)u})\left(\frac{\lambda}{\lambda+\mu}e^{-\mu w} + \frac{\mu}{\lambda+\mu}e^{-\lambda u}\right) = f(u)g(w)$$

and so since the joint distribution factors into a product of a function of u and a function of w the two r.v's are independent.

c) This follows from the fact that the r.v's are continuous.

d) Let $P_{Y|X+Y}(y|u)$ denote the conditional density of Y given $X+Y=u$.

$$\begin{aligned} p_{Y|X+Y}(y,u) &= \frac{p_{Y,X+Y}(y,u)}{p_{X+Y}(u)} \\ &= \frac{p_X(u-y)p_Y(y)}{\int_0^u p_X(u-x)P_Y(x)dx} \\ &= \frac{\lambda^2 e^{-\lambda u}}{\lambda^2 u e^{-\lambda u}} = \frac{1}{u} \end{aligned}$$

showing that the conditional distribution of Y given $X+Y=u$ is uniform in $[0,u]$.

5. We are given that X and Y are jointly Gaussian with 0 mean, unit variances and correlation ρ. Let Z denote the column vector $(X,Y)^T$

   Hence the characteristic function is given by

$$C_Z(h) = \mathbf{E}[e^{j[h,Z]} = e^{-\frac{1}{2}[Rh,h]}$$

   where $R_{1,1} = R_{2,2} = 1$ and $R_{1,2} = R_{2,1} = \rho$.

   Now let $W = col(X+Y, X-Y)$ then $W = AZ$ where A is a matrix given by:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   and hence the characteristic function of W is

$$C_W(h) = e^{-\frac{1}{2}[A^*RAh,h]}$$

   and A^*RA is the matrix given by:

$$A = \begin{bmatrix} 2(1+\rho) & 0 \\ 0 & 2(1-\rho) \end{bmatrix}$$

   that shows that $cov(X+Y, X-Y) = 0$ implying that $X+Y$ and $X-Y$ are independent given that X and Y are jointly Gaussian. Moreover we have that $X+Y$ is $N(0, 2(1+\rho))$ and $X-Y$ is $N(0, 2(1-\rho))$.

6. Let us find the distribution of Y. We need to consider 3 cases$Y \geq a$, $Y \leq -a$ and $-a \leq Y \leq a$

   For $y > a$

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \mathbb{P}(a < Y \leq y) + \mathbb{P}(-a \leq Y \leq a) + P(Y < -a) \\ &= P(a < -X \leq y) + P(-a < X \leq a) + P(X > a) \\ &= P(-a > X \geq -y) + P(-a < X \leq a) + P(X > a) \\ &= P(X \leq y) = \Phi_X(y) \end{aligned}$$

Similarly for $-a < y < a$ we have $\mathbb{P}(Y \le y) = \Phi_X(y)$ and for $y < -a$ we can also show that $\mathbb{P}(Y \le y) = \Phi_X(y)$ showing that Y has a $N(0,1)$ distribution.

Now let $\rho(a) = \mathbf{E}[XY]$ and $\phi(x)$ denote the N(0,1)density

$$\begin{aligned} \rho(a) = \mathbf{E}[XY] &= \int_{-a}^{a} x^2\phi(x)dx - \int_{-\infty}^{-a} x^2\phi(x)dx - \int_{a}^{\infty} x^2\phi(x)dx \\ &= 1 - 4\int_{a}^{\infty} x^2\phi(x)dx \end{aligned}$$

Let a^* be the root of $\rho(a) = 0$ i.e. $\int_a^\infty x^2\phi(x)dx = 0.25$ (it clearly exists since $\int_0^\infty x^2\phi(x)dx = 0.5$). Now if X and Y are jointly Gaussian it would imply that they are independent. But clearly this is not the case since for $a > 0$ $P(X > a, Y > a) \ne P(X > a)P(Y > a)$ as $P(X > a, Y > a) = P(X > a)$. Therefore X and Y are not jointly Gaussian even if they are individually Gaussian.

7. This problem has been done in the notes.

8. $X_{n+1} = \sum_{j=1}^{X_n} Z_n^{(j)}$.

   a) Let $m_n = \mathbf{E}[X_n]$ then using the result of Problem 7 we have:

   $$m_{n+1} = \mu m_n$$

   noting that $m_0 = 1$ we have $m_n = \mathbf{E}[X_n] = \mu^n$.

   b) Note that by the definition of a branching process conditioned on X_m the expected number of offspring are: $X_m\mu^{n-m}$ i.e.

   $$\mathbf{E}[X_n|X_m] = X_m\mu^{n-m}$$

   Therefore

   $$\mathbf{E}[X_nX_m] = \mathbf{E}[\mathbf{E}[X_nX_m|X_m]] = \mathbf{E}[X_mX_m\mu^{n-m}] = \mu^{n-m}\mathbf{E}[X_m^2]$$

   Hence:

   $$\mathbf{E}[(X_n - \mu^n)(X_m - \mu^m)] = E[X_nX_m] - \mu^{n+m} = \mu^{n-m}(\mathbf{E}[X_m^2] - \mu^{2m}) = \mu^{n-m}var(X_m)$$

   c) From the notes $P(X_{n+1} = 0) = g(P(X_n = 0))$ where $g(z) = \sum_{n=0}^\infty P(Z = n)z^n$ the moment generating function of Z_n. Now when $p = q$, $g(z) = \sum_{n=0}^\infty p^{n+1}z^n = \frac{p}{1-pz}$
   Therefore noting that $X_0 = 1$ we have

   $$P(X_n = 0) = g^{(n)}(0)$$

   Now when $p = q = \frac{1}{2}$, we see $g^{(2)}(0) = g(g(0)) = \frac{p}{1-p^2} = \frac{2}{3}$ and by induction

   $$g^{(n)}(0) = \frac{n}{n+1}$$

   When $p \ne q$ we have $g(z) = \sum_{n=0}^\infty qp^nz^n = \frac{1-p}{1-pz}$ from which we obtain that:

   $$g^{(n)}(0) = \frac{q(p^n - q^n)}{p^{n+1} - q^{n+1}}$$