ECE604- Stochastic Processes Problem Set # 3
Fall 2013

Due Monday Oct. 27, 2013

Do all problems. This problem set is essentially computational. You should get used to perform standard computations using Jacobians, conditional probabilities, expectations, working with characteristic functions and the like.

1. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed (i.i.d) random variables for which $E[X_1^{-1}]$ exists. Let $S_n = \sum_{i=1}^{n} X_i$. Show that for $m < n$

$$E[\frac{S_m}{S_n}] = \frac{m}{n}$$

Find $E[S_m|S_n]$ and $E[S_n|S_m]$ assuming that $E[X_i] = 1$.

2. Let X and Y be two jointly distributed r.v's with joint density:

$$p_{XY}(x, y) = \frac{1}{x}, \qquad 0 \le y \le x \le 1$$

Find

(a) $p_X(x)$ the marginal density of X

(b) Show that , given $\{X = x\}$, the r.v. Y is uniformly distributed on $[0, x]$ i.e.the conditional density:

$$p_{Y|X}(y|x) = \frac{1}{x} \quad y \in [0, x]$$

(c) Find $E[Y|X = x]$

(d) Calculate $\mathbf{P}(X^2 + Y^2 < 1|X = x)$ and hence show that $\mathbf{P}(X^2 + Y^2 \le 1) = \ln(1 + \sqrt{2})$

3. Find the density function of $Z = X + Y$ when X and Y have the joint density:

$$p_{X,Y}(x, y) = \frac{(x + y)}{2} e^{-(x+y)} \quad , \quad x, y \ \ge 0$$

4. Let X and Y be independent, exponentially distriibuted r.v's with parameters λ and μ respectively. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$ and $W = V - U$.

(a) Find $P(U = X) = P(X \le Y)$

(b) Show that U and W are independent.

(c) Show that $P(X = Y) = 0$

(d) Suppose that $\lambda = \mu$. Show that the conditional distribution of Y given that $X + Y = u$ is uniform in $[0, u]$.

(e) Now let $U = \min\{X_1, X_2, \ldots, X_n\}$ and $V = \max\{X_1, X_2, \ldots, X_n\}$ where $\{X_i\}_{i=1}^n$ are i.i.d.. Suppose $U = u$. Are U and V independent? Find $P(V|U = u)$.

5. Let X and Y be jointly Gaussian r.v's with 0 means and unit variances, and correlation ρ. Find the joint density function of $X+Y$ and $X-Y$, and their marginal density functions. What do you notice?

6. Let $X \sim N(0,1)$ and let $a > 0$. Show that the r.v. Y defined by:

$$\begin{aligned} Y &= X \quad if \quad |X| < a \\ &= -X \quad if \quad |X \geq a \end{aligned}$$

has a $N(0,1)$ distribution. Find an expression for $\rho(a) = cov(X,Y)$ in terms of the normal density function. Are the pair (X,Y) jointly Gaussian?

7. Let $X_1, X_2, \cdots$ denote a sequence of i.i.d. random variables. Let $N(\omega)$ denote an integer valued random variable that is independent of the sequence. Define:

$$Y = \sum_{k=1}^{N} X_k$$

a) The moment generating function of Y interms of the moment generating function of X denoted by $M_X(t)$

b) Find the mean and variance of Y in terms of the corresponding quantities of X.

c) Define:

$$I(a) = \max_{\theta}\{a\theta - \ln m_X(\theta)\}$$

where $M_X(\theta)$ is the moment generating function of X. Show that it is positive and achieves its maximum for $\theta \neq \mathbf{E}[\mathbf{X}]$. Now define the following probability distribution:

$$p(x) = \frac{e^{\theta_a x}}{M_X(\theta_a)} p_X(x)$$

where $p_X(.)$ is the density of X and θ_a is the point where $I(a)$ achieves its maximum. If Y is a random variable that has $p(.)$ as its density, find its mean and variance.

8. Let X_n be the number of individuals of the n-th generation of a Galton-Watson branching process defined as:

$$X_{n+1} = \sum_{j=1}^{X_n} Z_n^{(j)}$$

where $Z_k^{(i)}$ are i.i.d. with mean μ.

(a) Show that $E[X_n] = \mu^n$

(b) Show that the covariance $E[(X_n - \mu^n)(X_m - \mu^m)] = \mu^{n-m}(E[X_m^2] - \mu^{2m})$ when $m \leq n$.

(c) Suppose the r.v's of offsprings $Z_k^{(i)}$ are geometrically distributed with $P(Z_i = k) = qp^k$. where $p + q = 1$. Show that:

$$\begin{aligned} P(X_n = 0) &= \frac{n}{n+1}, \quad p = q = 0.5 \\ &= \frac{q(p^n - q^n)}{p^{n+1} - q^{n+1}}, \quad p \neq q \end{aligned}$$