Topics in Electrical and Computer Engineering

Phasors

Douglas Wilhelm Harder
Department of Electrical and Computer Engineering
University of Waterloo

Phasors
Outline

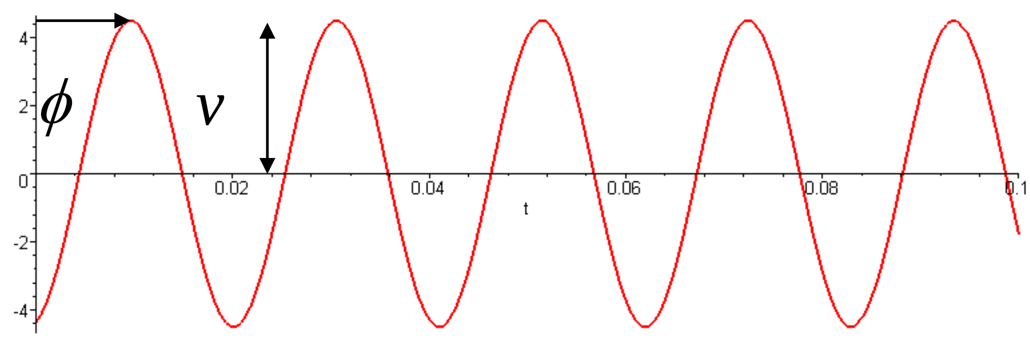
- In this topic, we will look at
  - The necessary background
  - Sums of sinusoidal functions
  - The trigonometry involved
  - Phasor representation of sinusoids
  - Phasor addition
  - Why phasors work
  - Phasor multiplication and inverses
  - Phasors for circuits

Phasors
Background

- Consider any periodic sinusoid with
  - Period $T = 1/f = 2\pi/\omega$
  - Frequency $f = 1/T = \omega/2\pi$
  - Angular frequency $\omega = 2\pi f = 2\pi/T$
- It is possible to write such a sinusoid as

$$v \cos(\omega t + \phi)$$

where
  - v is the *amplitude*
  - ϕ is the *phase shift*



Phasors

Background

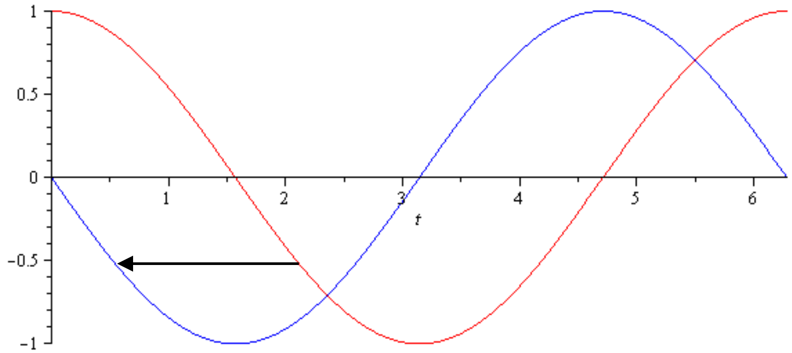
- As $v\cos(\omega t + \phi + 2\pi) = v\cos(\omega t + \phi)$, restrict $-\pi < \phi \leq \pi$
- Engineers throw an interesting twist into this formulation
  - The frequency term ωt has units of radians
  - The phase shift ϕ has units of degrees: $-180° < \phi \leq 180°$
- For example,

$$V\cos(377t + 45°)$$

$$V\cos(377t - 90°)$$

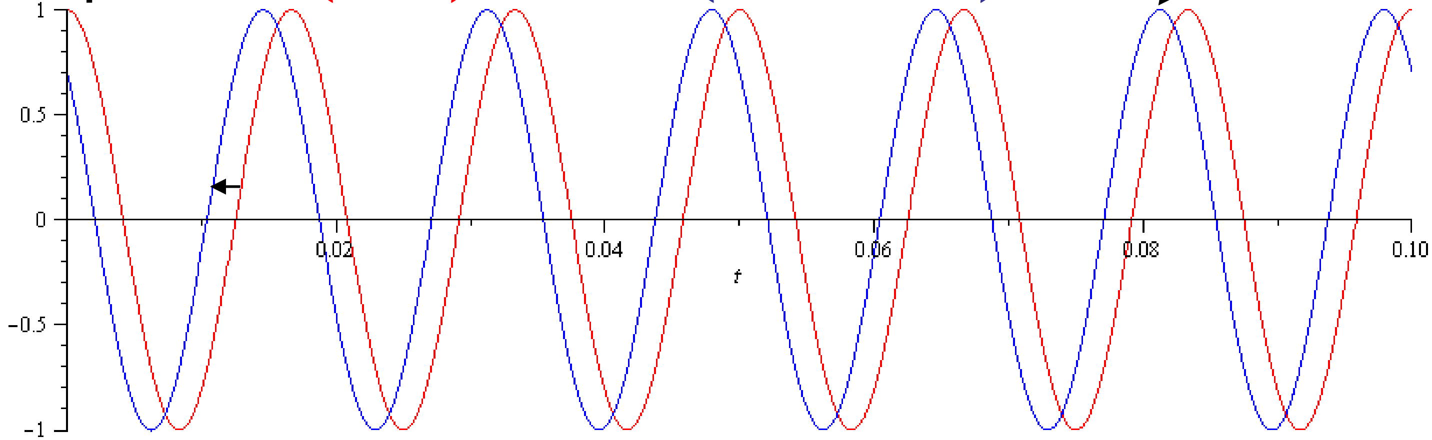
Phasors
Background

- A positive phase shift causes the function to *lead* of ϕ
- For example, $-\sin(t) = \cos(t + 90^\circ)$ leads $\cos(t)$ by 90°



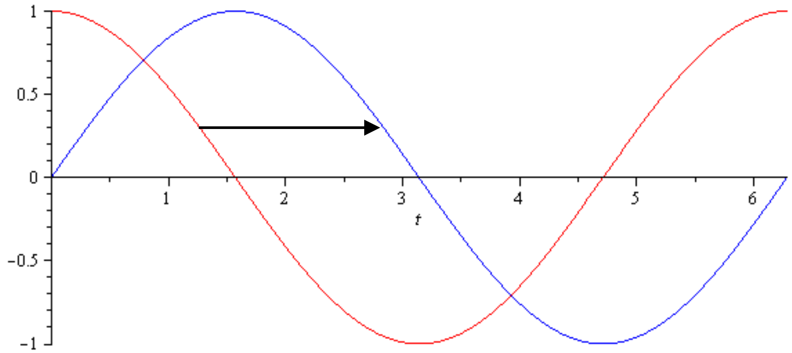
- Compare $\cos(377t)$ and $\cos(377t + 45^\circ)$

Blue peaks first
- Blue is leading

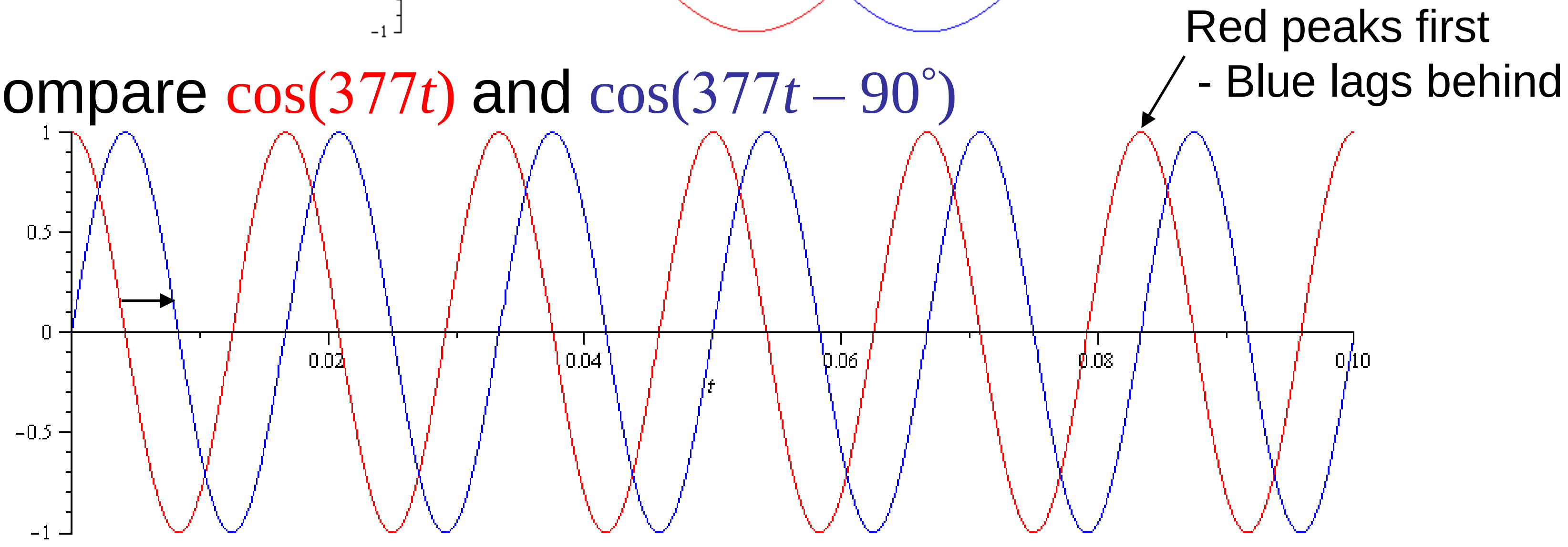


Phasors
Background

- A negative phase shift causes the function to *lag* by ϕ
- For example, $\sin(t) = \cos(t - 90^\circ)$ lags $\cos(t)$ by 90°

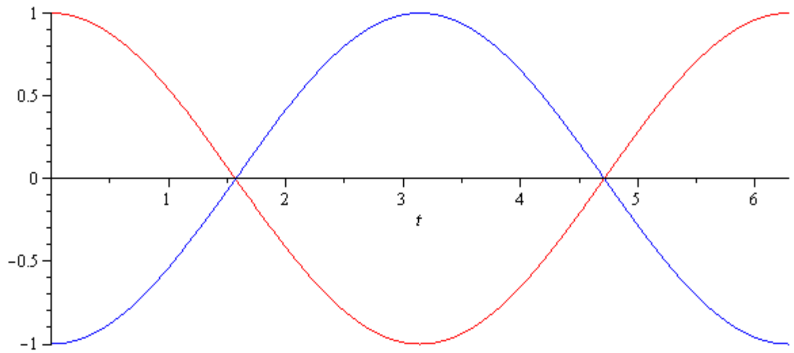


- Compare $\cos(377t)$ and $\cos(377t - 90^\circ)$

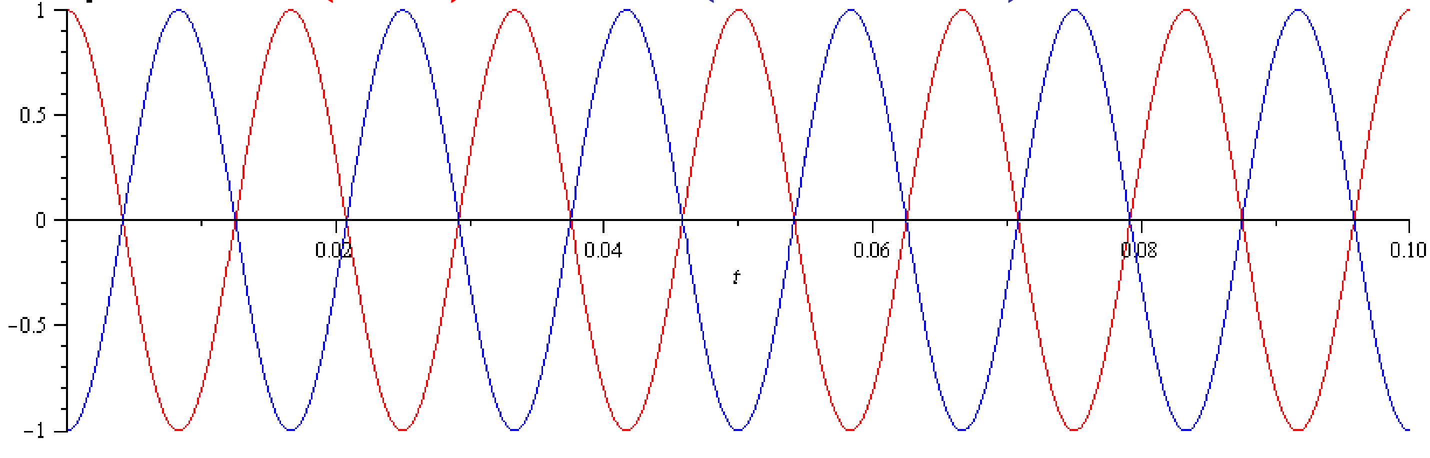


Phasors
Background

- If the phase shift is 180°, the functions are *out of phase*
- *E.g.*, $-\cos(t) = \cos(t - 180^\circ)$ and $\cos(t)$ are out of phase



- Compare $\cos(377t)$ and $\cos(377t - 180^\circ)$



Phasors
Background

- Suppose we add a number of sinusoidal voltages with:
  - Equal frequencies
  - Different amplitudes (voltages) and phase shifts
- *E.g.*, $3\cos(377t+75°)+10\cos(377t+50°)+4\cos(377t-15°)+$
  $5\cos(377t+42°)+12\cos(377t-90°)$

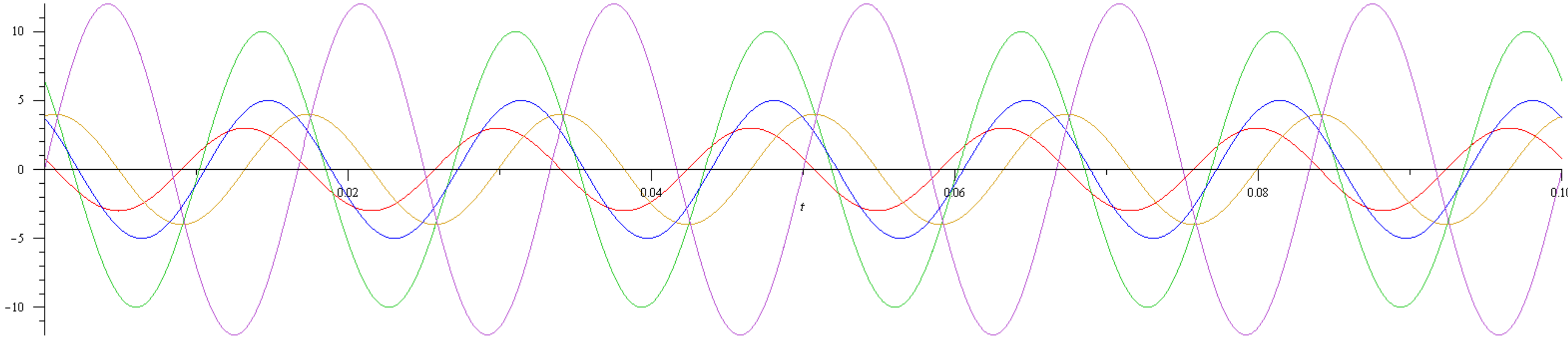
- It may not be obvious, but the result will be another sinusoid of the form

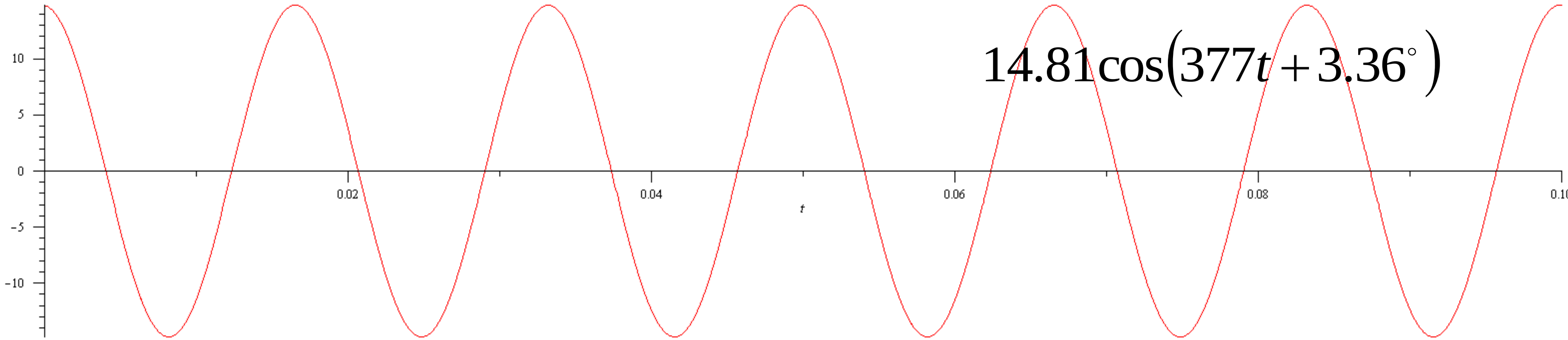
$$A\cos(377t+\phi)$$

Phasors

Background

- These are the five sinusoids and their sum





$$14.81\cos\left(377t + 3.36^{\circ}\right)$$

Phasors
Derivations (The Hard Way)

- We will now show that the sum of two sinusoids with
  - The same frequency, and
  - Possibly different amplitudes and phase shifts

  is a sinusoid with the same frequency
- Our derivation will use trigonometric formula familiar to all high-school students
  - Later, we will see how exponentials with complex powers simplify this observation!

Phasors
Example Derivation (The Hard Way)

- Consider the sum of two sinusoids:

$$3\cos(\omega t)+9\sqrt{2}\cos\left(\omega t+45^\circ\right)$$

- Use the rule

$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$

- Thus we expand the terms

$$9\sqrt{2}\cos\left(\omega t+\tfrac{\pi}{2}\right)=9\sqrt{2}\cos(\omega t)\cos\left(45^\circ\right)-9\sqrt{2}\sin(\omega t)\sin\left(45^\circ\right)$$

$$=9\cos(\omega t)-9\sin(\omega t)$$

Example Derivation (The Hard Way)

- Therefore

$$3\cos(\omega t)+9\sqrt{2}\cos(\omega t+45^\circ)=12\cos(\omega t)-9\sin(\omega t)$$

- We wish to write this as

$$v\cos(\omega t+\phi)=v\cos(\omega t)\cos\phi-v\sin(\omega t)\sin\phi$$

- We therefore deduce that

$$v\cos\phi=12$$

$$v\sin\phi=9$$

Phasors

Example Derivation (The Hard Way)

- Given

$$v\cos\phi = 12$$

$$v\sin\phi = 9$$

- Square both sides and add:

$$(v\cos\phi)^2 + (v\sin\phi)^2 = 12^2 + 9^2$$

$$v^2(\cos^2\phi + \sin^2\phi) = 225$$

- Because $\cos^2\phi + \sin^2\phi = 1$ it follows that

$$v = \sqrt{225} = 15$$

Phasors

Example Derivation (The Hard Way)

- Again, given

$$v\cos\phi = 12$$

$$v\sin\phi = 9$$

- Take the ratio:

$$\frac{v\sin\phi}{v\cos\phi} = \tan\phi = \frac{9}{12} = 0.75$$

- Therefore $\phi \approx 36.87^\circ$

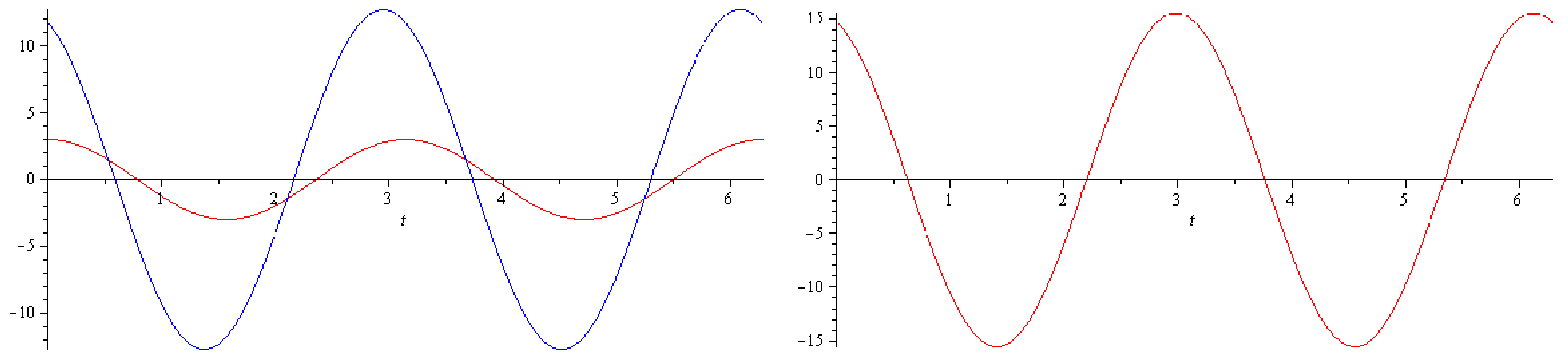
Phasors
Example Derivation (The Hard Way)

- It follows that

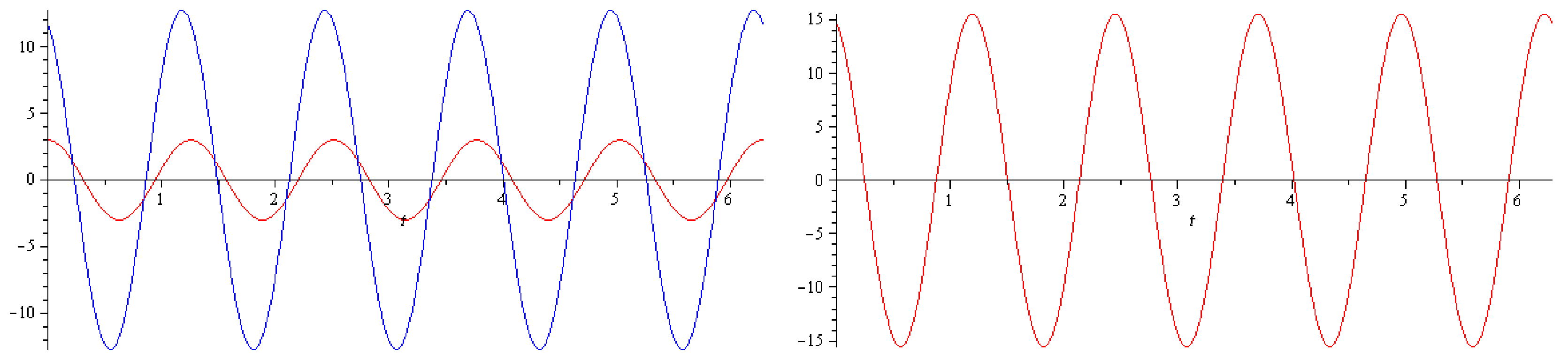
$$3\cos(\omega t)+9\sqrt{2}\cos(\omega t+45^\circ)=15\cos(\omega t+36.87^\circ)$$

- This is independent of the frequency:

$\omega = 2$



$\omega = 5$



Phasors

Full Derivation (The Hard Way)

- Consider the sum of two sinusoids:

$$v_1 \cos(\omega t + \phi_1) + v_2 \cos(\omega t + \phi_2)$$

- Using the rule

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

we can expand

$$\begin{aligned}
&v_1 \cos(\omega t + \varphi_1) + v_2 \cos(\omega t + \varphi_2) \\
&\quad = v_1 \cos(\omega t)\cos(\varphi_1) - v_1 \sin(\omega t)\sin(\varphi_1) + \\
&\qquad v_2 \cos(\omega t)\cos(\varphi_2) - v_2 \sin(\omega t)\sin(\varphi_2) \\
&\quad = \cos(\omega t)\big(v_1 \cos(\varphi_1) + v_2 \cos(\varphi_2)\big) - \\
&\qquad \sin(\omega t)\big(v_1 \sin(\varphi_1) + v_2 \sin(\varphi_2)\big)
\end{aligned}$$

Phasors

Full Derivation (The Hard Way)

- Now, given

$$v_1\cos(\omega t+\phi_1)+v_2\cos(\omega t+\phi_2)$$
$$=\cos(\omega t)(v_1\cos(\phi_1)+v_2\cos(\phi_2))-\sin(\omega t)(v_1\sin(\phi_1)+v_2\sin(\phi_2))$$

Suppose we can write this in the form

$$v\cos(\omega t+\phi)$$

- First, $v\cos(\omega t+\phi)=v\cos(\omega t)\cos\phi-v\sin(\omega t)\sin\phi$

and thus
$$v\cos(\phi)=v_1\cos(\phi_1)+v_2\cos(\phi_2)$$
$$v\sin(\phi)=v_1\sin(\phi_1)+v_2\sin(\phi_2)$$

Phasors

Full Derivation (The Hard Way)

- First, given

$$v\cos(\phi) = v_1\cos(\phi_1) + v_2\cos(\phi_2)$$
$$v\sin(\phi) = v_1\sin(\phi_1) + v_2\sin(\phi_2)$$

square both sides and add

$$v^2 = {v_1}^2 + 2v_1v_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + {v_2}^2$$

$$\therefore v = \sqrt{{v_1}^2 + 2v_1v_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + {v_2}^2}$$

Phasors

Full Derivation (The Hard Way)

- Similarly, given

$$v\cos(\phi) = v_1\cos(\phi_1) + v_2\cos(\phi_2)$$
$$v\sin(\phi) = v_1\sin(\phi_1) + v_2\sin(\phi_2)$$

take the ratio

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{v_1\sin\phi_1 + v_2\sin\phi_2}{v_1\cos\phi_1 + v_2\cos\phi_2}$$

$$\therefore \phi = \tan^{-1}\left(\frac{v_1\sin\phi_1 + v_2\sin\phi_2}{v_1\cos\phi_1 + v_2\cos\phi_2}\right)$$

Phasors

Full Derivation (The Hard Way)

- Adding n sinusoids must be done one pair at a time:

$$\underbrace{v_1 \cos(\omega t + \phi_1) + v_2 \cos(\omega t + \phi_2)} + v_3 \cos(\omega t + \phi_3) + \cdots$$

$$\underbrace{v_{1,2} \cos(\omega t + \phi_{1,2}) \qquad + v_3 \cos(\omega t + \phi_3)} + \cdots$$

$$v_{1,2,3} \cos(\omega t + \phi_{1,2,3}) \qquad + \cdots$$

Phasors

Using Phasors

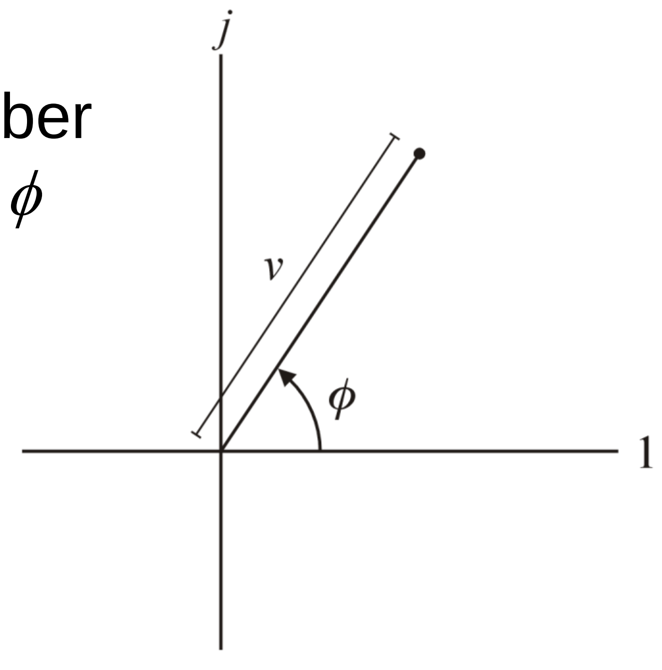
- Rather than dealing with trigonometric identities, there is a more useful representation: *phasors*
- Assuming ω is fixed, associate

$$v\cos(\omega t+\phi) \Leftrightarrow v\angle\phi$$

where $v\angle\phi$ is the complex number with magnitude v and argument ϕ

- The value $v\angle\phi$ is a phasor and is read as

"vee phase fee"



Using Phasors

- The addition of sinusoids is equivalent to phasor addition

Function of Time | Phasors

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\sum_{k=1}^{n} v_k \cos(\omega t + \phi_k) \arrow[r] \arrow[d, "\text{trigonometry}"'] & \sum_{k=1}^{n} v_k \angle \phi_k \arrow[d, "\text{complex number addition}"] \\
v \cos(\omega t + \phi) & v \angle \phi \arrow[l]
\end{tikzcd}$$

- We *transform* the problem of trigonometric addition into a simpler problem of complex addition

Phasors

Using Phasors

- Recalling Euler's identity

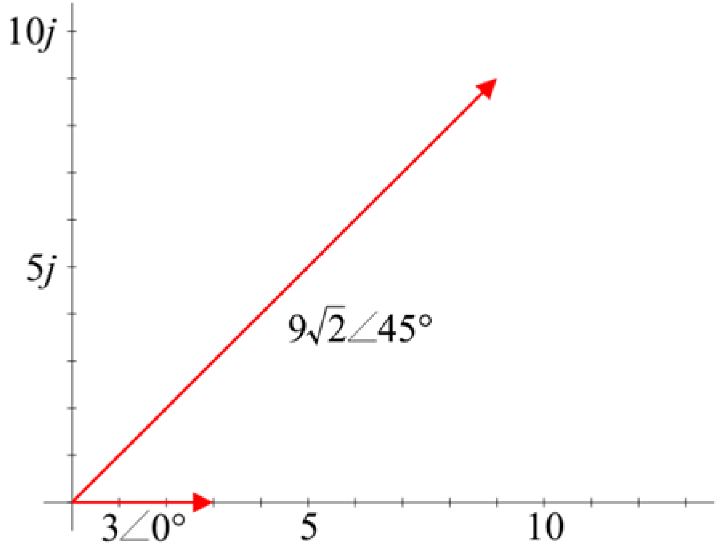
$$v\angle\phi = ve^{j\phi} = v(\cos\phi + j\sin\phi)$$

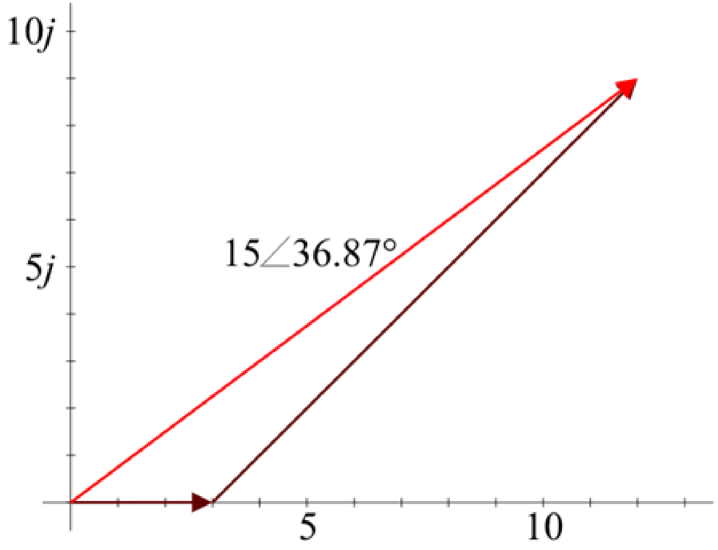
- Therefore

$$\sum_{k=1}^{n} v_k \angle\phi_k = \sum_{k=1}^{n} v_k \cos\phi_k + j\sum_{k=1}^{n} v_k \sin\phi_k$$

Phasors
Using Phasors

- Given our motivating example $3\cos(\omega t)+9\sqrt{2}\cos(\omega t+45^\circ)$
  - Using phasors: $3\angle 0^\circ + 9\sqrt{2}\angle 45^\circ = 3+9+9j$
    $$= 12+9j$$
    $$= 15\angle 36.87^\circ$$
  - The result is $15\cos(\omega t+36.87^\circ)$





Using Phasors

- For example, given our first sum:

$$3\cos(377t+75°)+10\cos(377t+50°)+4\cos(377t-15°)+$$
$$5\cos(377t+42°)+12\cos(377t-90°)$$

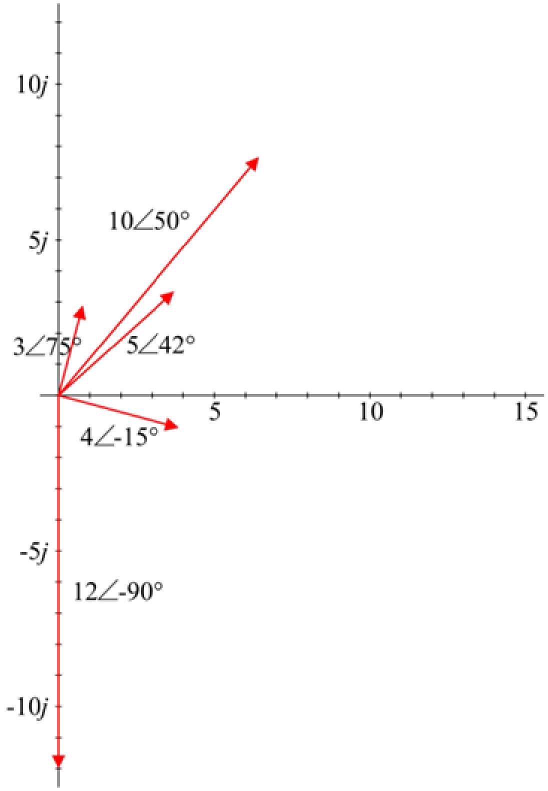
- Using phasors

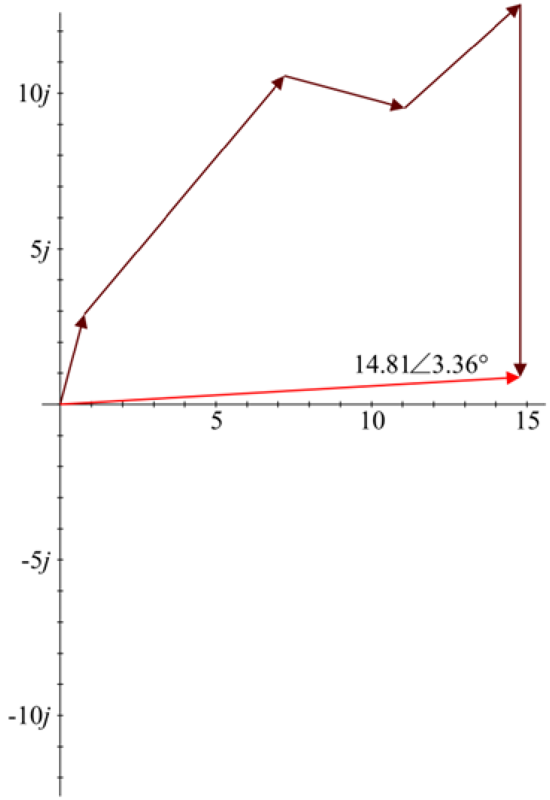
$$3\angle 75^\circ+10\angle 50^\circ+4\angle\left(-15^\circ\right)+5\angle 42^\circ+12\angle\left(-90^\circ\right)$$
$$=3e^{75^\circ j}+10e^{50^\circ j}+4e^{-15^\circ j}+5e^{42^\circ j}+3e^{-90^\circ j}$$
$$=14.78+0.87j$$
$$=14.81\angle 3.36^\circ$$

we get the sum $14.81\cos\left(377t+3.36^\circ\right)$

Using Phasors

- These graphs show:
  - The individual phasors
  - The sum of the phasors

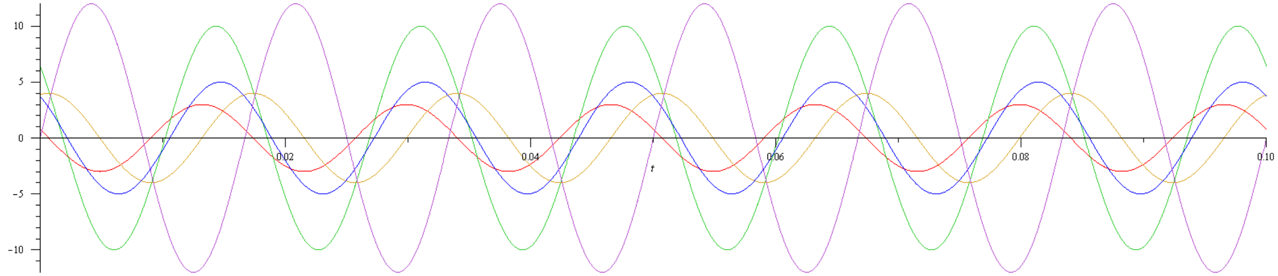




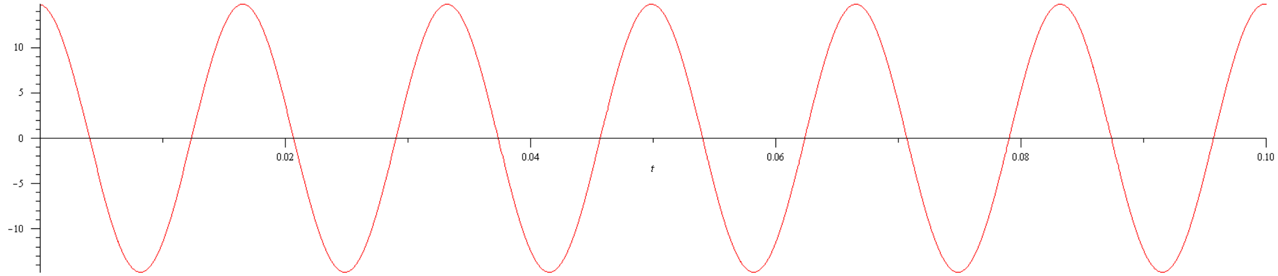
Phasors
Using Phasors

- A plot of the sum

$$3\cos(377t+75°)+10\cos(377t+50°)+4\cos(377t-15°)+$$
$$5\cos(377t+42°)+12\cos(377t-90°)$$



is identical to a plot of $14.81\cos(377t+3.36^{\circ})$



Phasors
Why Phasors Work

- To understand recall that

$$ve^{j(\omega t+\phi)} = v\cos(\omega t+\phi) + jv\sin(\omega t+\phi)$$

- Therefore $v\cos(\omega t+\phi) = \Re\left(ve^{j(\omega t+\phi)}\right)$
- Plot the complex exponential

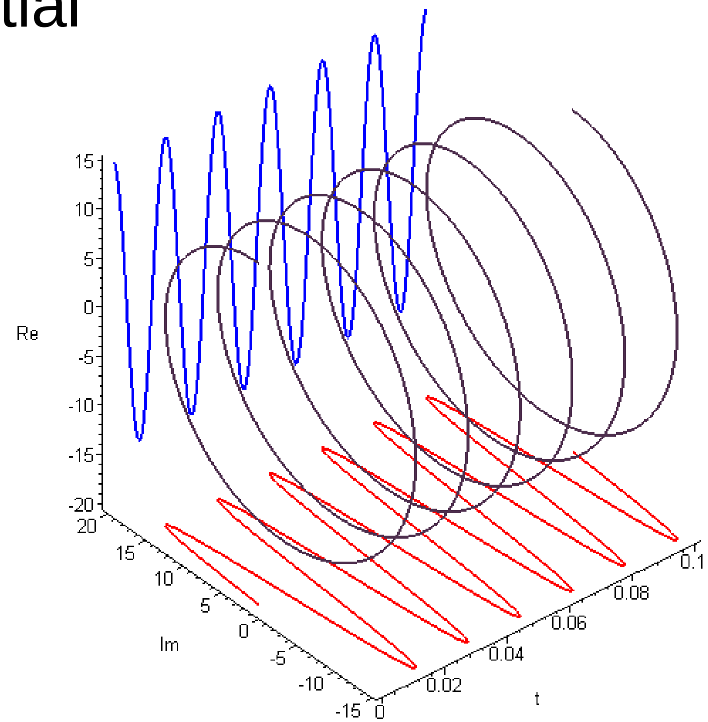
$$14.81e^{j\left(377t+3.36^\circ\right)}$$

The real part:

$$14.81\cos(377t + 3.36^\circ)$$

The imaginary part:

$$14.81\sin(377t + 3.36^\circ)$$



Phasors

Why Phasors Work

- Note that

$$v_1 e^{j(\omega t+\phi_1)} + v_2 e^{j(\omega t+\phi_2)} = v_1 e^{j\omega t} e^{j\phi_1} + v_2 e^{j\omega t} e^{j\phi_2}$$
$$= \left(v_1 e^{j\phi_1} + v_2 e^{j\phi_2}\right) e^{j\omega t}$$
$$= \left(v_1 \angle \phi_1 + v_2 \angle \phi_2\right) e^{j\omega t}$$

and because $\Re\{w+z\} = \Re\{w\} + \Re\{z\}$, it follows

$$v_1 \cos(\omega t + \phi_1) + v_2 \cos(\omega t + \phi_2) = \Re\left\{v_1 e^{j(\omega t+\phi_1)}\right\} + \Re\left\{v_2 e^{j(\omega t+\phi_2)}\right\}$$
$$= \Re\left\{v_1 e^{j\omega t} e^{j\phi_1}\right\} + \Re\left\{v_2 e^{j\omega t} e^{j\phi_2}\right\}$$
$$= \Re\left\{v_1 e^{j\omega t} e^{j\phi_1} + v_2 e^{j\omega t} e^{j\phi_2}\right\}$$
$$= \Re\left\{\left(v_1 \angle \phi_1 + v_2 \angle \phi_2\right) e^{j\omega t}\right\}$$

Phasors
Derivation (The Easy Way)

- Previously, we used trigonometry to show that the sum of two sinusoids with the same frequency must continue to have the same frequency
- Using complex exponentials, this becomes obvious

$$\Re\left\{v_1 e^{j(\omega t+\phi_1)}\right\}+\Re\left\{v_2 e^{j(\omega t+\phi_2)}\right\}=\Re\left\{\left(v_1\angle\phi_1+v_2\angle\phi_2\right)e^{j\omega t}\right\}$$

Phasors
Phasor Multiplication

- Multiplication may be performed directly on phasors:
  - The product of two phasors $r_1\angle\phi_1$ and $r_2\angle\phi_2$ is the phasor

$$(r_1 r_2)\angle(\phi_1+\phi_2)$$

  - For example,

$$(5\angle 45^\circ)\cdot(2\angle -60^\circ)=10\angle -15^\circ$$

  - This generalizes to products of n phasors:

$$(r_1\angle\phi_1)\cdots(r_n\angle\phi_n)=(r_1\cdots r_n)\angle(\phi_1+\cdots+\phi_n)$$

Phasor Multiplication

- Calculating the inverse is also straight-forward:
  - The inverse of the phasor $r\angle\phi$ is the phasor

$$\left(\frac{1}{r}\right)\angle(-\phi)$$

  - For example,

$$\left(5\angle 45^\circ\right)^{-1} = 0.2\angle -45^\circ$$

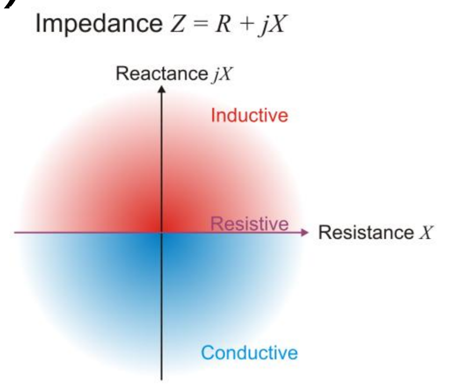
  - Using multiplication, we note the product is 1:

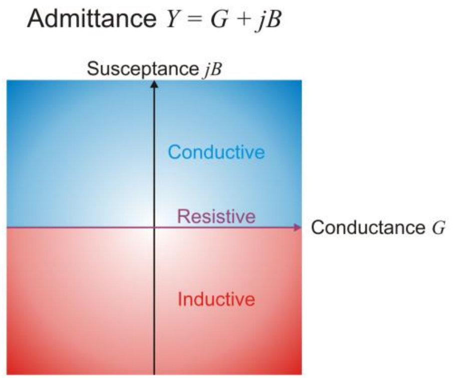
$$(r\angle\phi)\cdot\left(\left(\frac{1}{r}\right)\angle(-\phi)\right) = \left(r\frac{1}{r}\right)\angle(\phi+(-\phi)) = 1\angle 0^\circ$$

Phasors
Linear Circuit Elements

- When dealing with alternating currents (AC):
  - The generalization of the resistance R is the complex impedance $Z = R + jX$
  - The generalization of the conductance G is the complex admittance $Y = G + jB$
- Ohm's law easily generalizes:

$$V = IZ$$





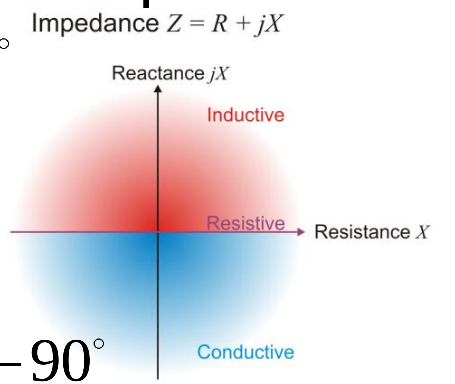
Phasors
Linear Circuit Elements

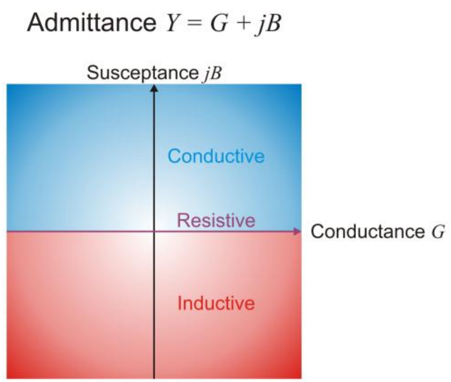
- With AC, the linear circuit elements behave like phasors

Inductor with inductance L $$Z = j\omega L = \omega L \angle 90^\circ$$

Resistor with resistance R $$Z = R\angle 0^\circ$$

Capacitor with capacitance C $$Z = \frac{1}{j\omega C} = \frac{1}{\omega C}\angle -90^\circ$$



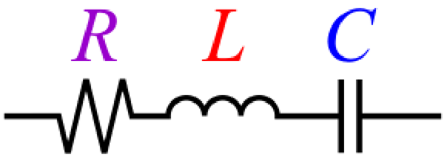


  - Note that inductance and capacitance are frequency dependant

- We can use this to determine the voltage across an linear circuit with AC: $V = IZ$

Phasors
Linear Circuit Elements

- Consider three circuit elements in series:



with

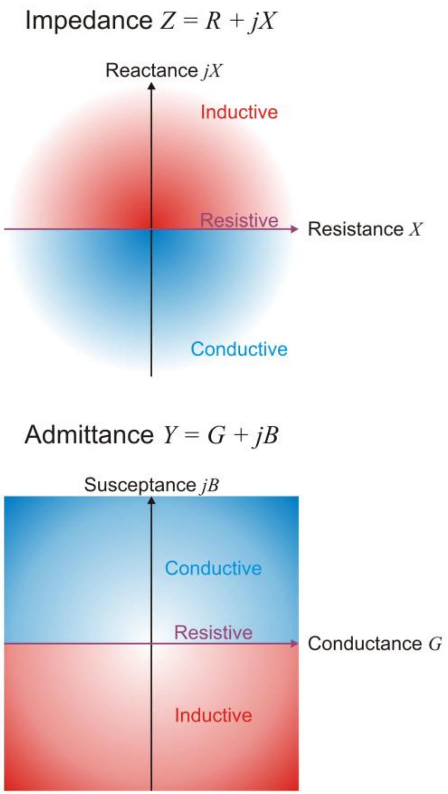
$$R = \quad 2\ \Omega$$
$$L = \quad 50\ \text{mH}$$
$$C = 750\ \text{mF}$$

- The total impedance is:

$$Z = 2\angle 0^\circ + \omega 0.050\angle 90^\circ + \frac{1}{0.750\omega}\angle -90^\circ$$

- *E.g.*, if $\omega = 1$, it follows that

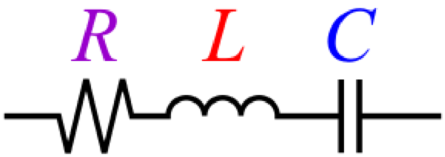
$$Z = 2.376\angle -32.69^\circ$$



Phasors

Linear Circuit Elements

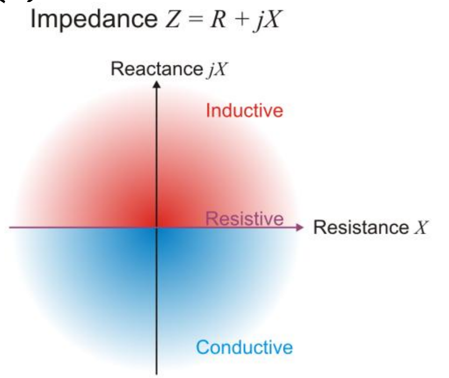
- If the current across this circuit is $I = 10\cos(t)$ A,

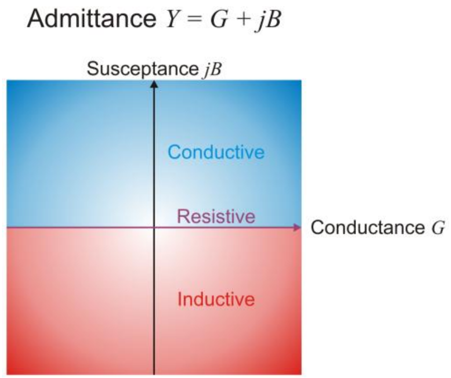


we may compute the voltage:

$$\begin{aligned} V &= IZ \\ &= \left(10\angle 0^{\circ}\right)\times\left(2.376\angle -32.69^{\circ}\right) \\ &= 23.76\angle -32.69^{\circ} \end{aligned}$$

or $V = 23.76\cos(t - 32.69^{\circ})$ V

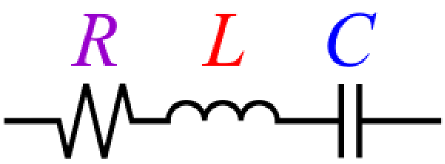


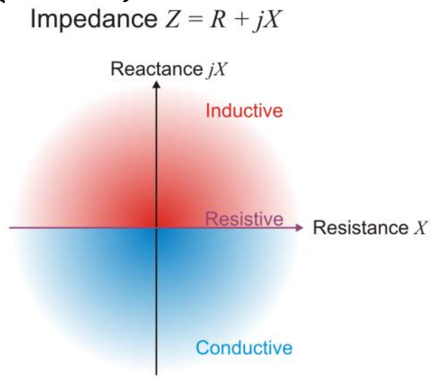


Phasors

Linear Circuit Elements

- If the current across this circuit is $I = 10\cos(377t)$ A,

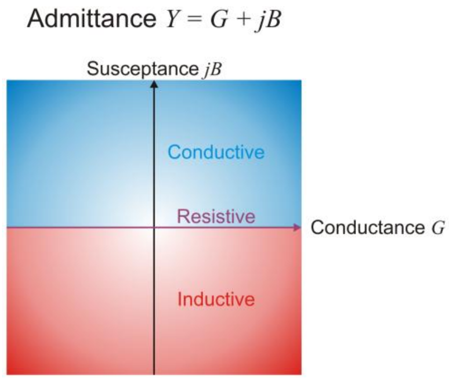




we need to recalculate the impedance:

$$Z = 2\angle 0^\circ + 377\cdot 0.050\angle 90^\circ + \frac{1}{0.750\cdot 377}\angle -90^\circ$$

$$= 18.95\angle 83.94^\circ$$



- Now $V = IZ$

$$= \left(10\angle 0^\circ\right)\times\left(18.95\angle 83.94^\circ\right)$$

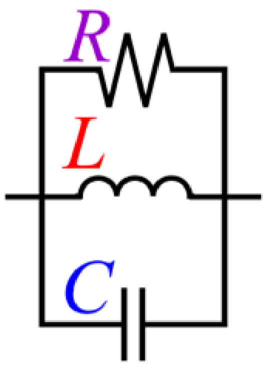
$$= 189.5\angle 83.94^\circ$$

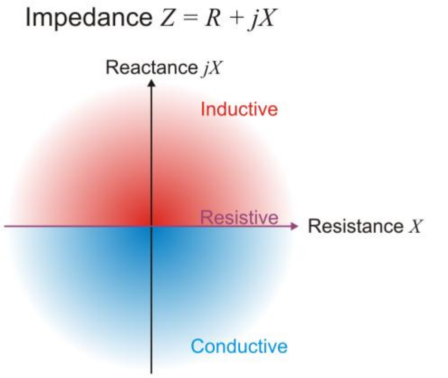
or $V = 189.5\cos(377t + 83.94^\circ)$ V

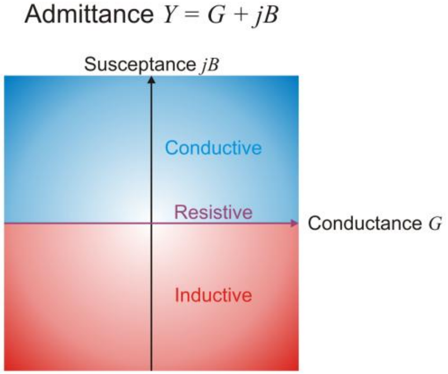
Phasors
Linear Circuit Elements

- Consider three circuit elements in parallel with

$$R = \ 2\ \Omega$$
$$L = \ 50\ \text{mH}$$
$$C = 750\ \text{mF}$$







- The total admittance is:

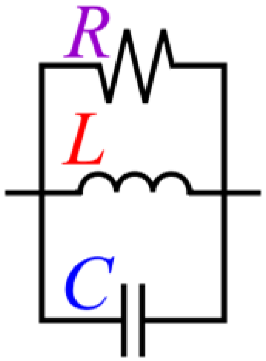
$$Y = \frac{1}{Z} = \frac{1}{2\angle 0^\circ} + \frac{1}{\omega 0.050\angle 90^\circ} + \frac{1}{\frac{\angle -90^\circ}{\omega 0.750}}$$

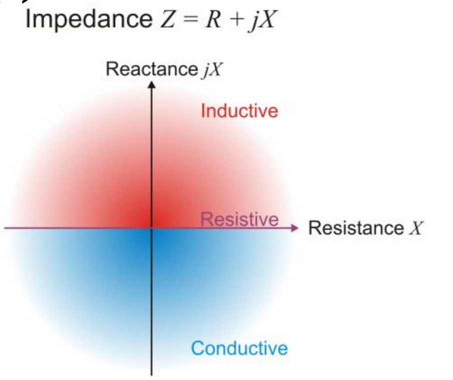
- *E.g.*, if $\omega = 1$, it follows that

$$Y = 0.5\angle 0^\circ + 20\angle -90^\circ + 0.75\angle 90^\circ$$
$$= 18.95\angle -83.94^\circ$$
$$\therefore Z = 0.05276\angle 83.94^\circ$$

Phasors
Linear Circuit Elements

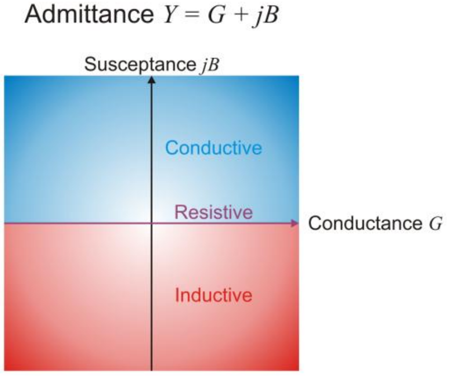
- If the current across this circuit is $I = 10\cos(t)$ A





we may compute the voltage:

$$\begin{aligned} V &= IZ \\ &= \left(10\angle 0^\circ\right)\times\left(0.05276\angle 83.94^\circ\right) \\ &= 0.5276\angle 83.94^\circ \end{aligned}$$

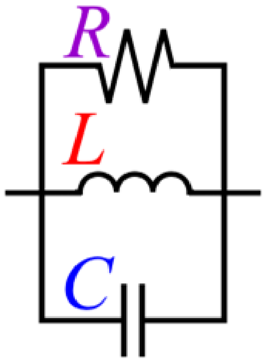


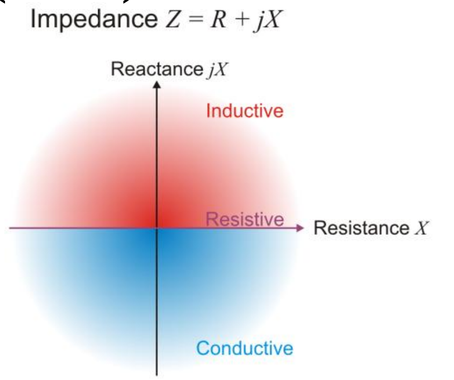
or $V = 0.5276\cos(t + 83.94^\circ)$ V

Phasors

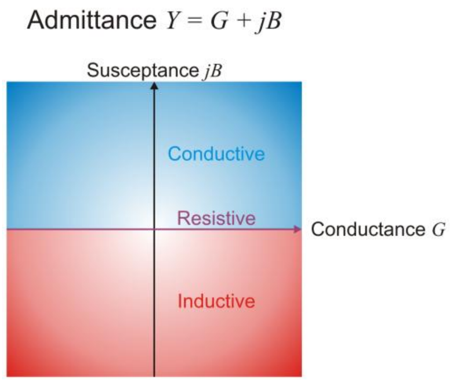
Linear Circuit Elements

- If the current across this circuit is $I = 10\cos(377t)$ A





we need to recalculate the impedance:



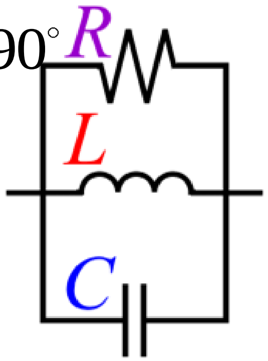
$$Y = \frac{1}{Z} = \frac{1}{2\angle 0^\circ} + \frac{1}{377 \cdot 0.050\angle 90^\circ} + \frac{1}{\frac{1}{377 \cdot 0.750}\angle -90^\circ}$$

$$= 282.7\angle 89.90^\circ$$

or $Z = 0.003537\angle -89.90^\circ$

Phasors
Linear Circuit Elements

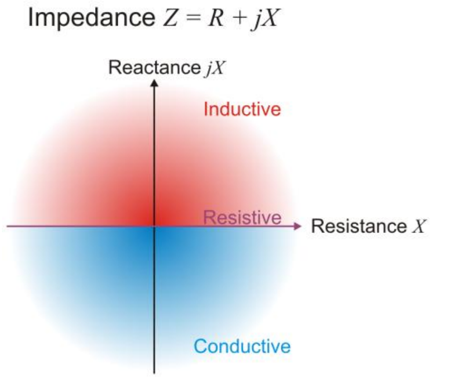
- Thus, with the current $I = 10\cos(377t)$ A and impedance $Z = 0.003537\angle -89.90^\circ$ we may now calculate

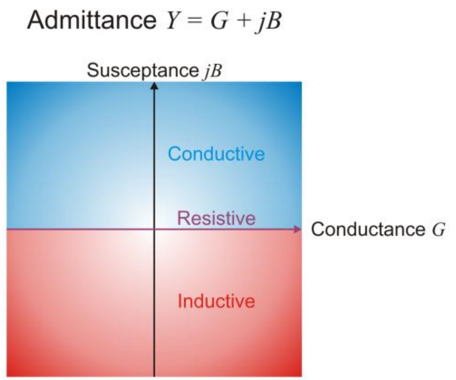


$$V = IZ$$
$$= \left(10\angle 0^\circ\right)\times\left(0.003537\angle -89.90^\circ\right)$$
$$= 0.03537\angle -89.90^\circ$$

or $V = 0.03537\cos(377t - 89.90^\circ)$ V

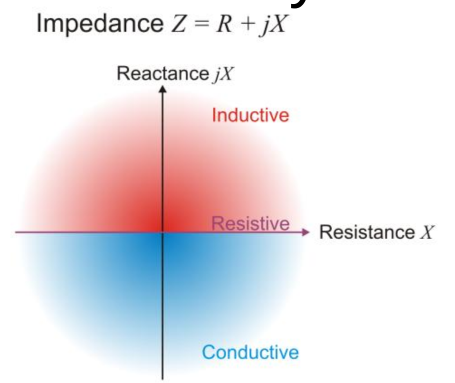
- This should be intuitive, as the capacitor acts as essentially a short-circuit for high frequency AC

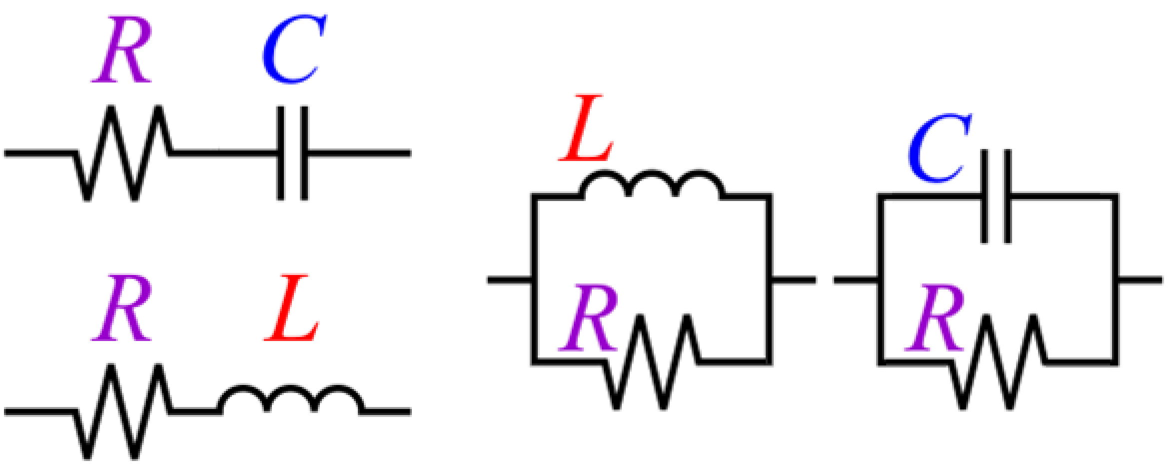


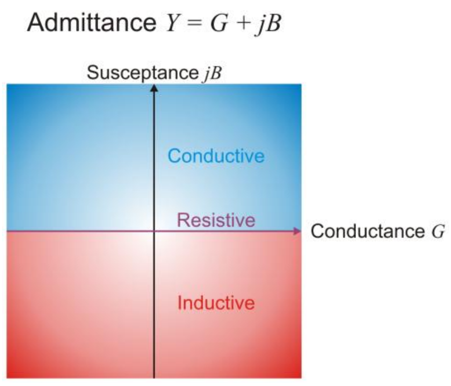


Phasors
Linear Circuit Elements

- A consequence of what we have seen here is that any linear circuit with alternating current may be replaced with
  - A single resistor, and
  - A single complex impedance
    - A capacitor or an inductor







Phasors
Final Observations

- These techniques do **not** work if the frequencies differ
- The choice of 377 in the examples is deliberate:
  - North American power is supplied at $60\ \mathrm{Hz}$:

$$2\pi 60 \approx 376.99111843$$

  - Relative error: 0.0023%
  - European power is supplied at $50\ \mathrm{Hz}$:

$$2\pi 50 = 100\pi \approx 314.159265358$$

Phasors
Final Observations

- Mathematicians may note the peculiar mix of radians and degrees in the formula

$$10\cos(377t + 50°)$$

Justification:

  - The phase shifts are 180° or less (π radians or less)
  - Which shifts are easier to visualize?

120°	2.094 rad or $2\pi/3$ rad
60°	1.047 rad or $\pi/3$ rad
45°	0.785 rad or $\pi/4$ rad
30°	0.524 rad or $\pi/6$ rad

  - Is it obvious that 2.094 rad and 0.524 rad are orthogonal?
  - Three-phase power requires the third roots of unity:
    - These cannot be written easily with radians (even using π)

Phasors
Summary

- In this topic, we will look at
  - Sum of sinusoidal functions
  - The trigonometry is exceptionally tedious
  - The phasor representation $v\cos(\omega t + \phi) \Leftrightarrow v\angle\phi$
  - The parallel between phasor addition and trigonometric summations
  - Phasor multiplication and inverses
  - Use of phasors with linear circuit elements

Phasors

Acknowledgments

- I would like to thank Prof. David Nairn for assistance in answering questions and Hua Qiang Cheng for pointing out errors in a previous version of these slides

Usage Notes

- These slides are made publicly available on the web for anyone to use
- If you choose to use them, or a part thereof, for a course at another institution, I ask only three things:
  - that you inform me that you are using the slides,
  - that you acknowledge my work, and
  - that you alert me of any mistakes which I made or changes which you make, and allow me the option of incorporating such changes (with an acknowledgment) in my set of slides

Sincerely,

Douglas Wilhelm Harder, MMath

dwharder@alumni.uwaterloo.ca