UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
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NE 112 *Linear algebra for nanotechnology engineering*

4.3 Unit vectors and normalization

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Introduction

- In this topic, we will
  - Define a *unit vector* for a given norm
  - Describe the algorithm for normalizing a vector
  - Introduce the hat notation for finite-dimensional unit vectors

Norms of vectors

- If $\mathbf{u} \in \mathbf{F}^n$, we will say that $\mathbf{u}$ is a *unit vector* if

$$\|\mathbf{u}\| = 1$$

  - This depends on the norm being used
- In this course, we will use the 2-norm

- If a finite-dimensional vector $\mathbf{u}$ is known to be a unit vector, we will put a “hat” on top of it: $\|\hat{\mathbf{u}}\| = 1$

  - Note that this differs for other vector spaces where a hat meant that the expression under it describes the vector in question

$$0.5^n = (1, 0.5, 0.25, 0.125, \ldots)$$

$$x^2$$

Norms of vectors

- If we are using the1-norm, then **u** is a unit vector when

$$|u_1| + |u_2| + \cdots + |u_n| = 1$$

- If we are using the ∞-norm, the **u** is a unit vector when

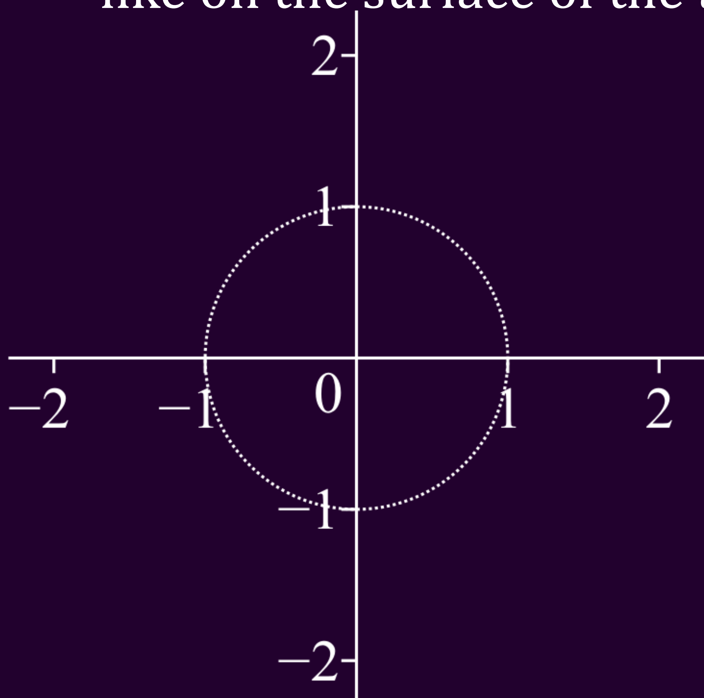
$$\max_{k=1,\ldots,n} \{|u_k|\} = 1$$

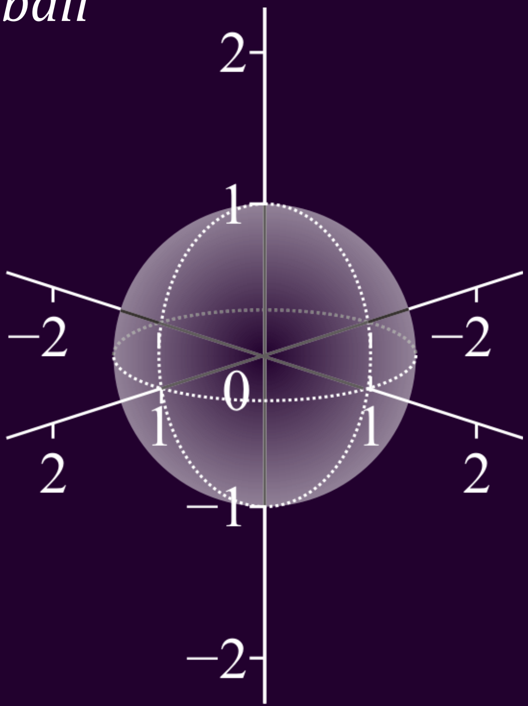
- If we are using the 2-norm, the **u** is a unit vector when

$$|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2 = 1$$

Norms of vectors

- For the 2-norm:
  - In $\mathbf{R}^2$, all unit vectors lie on the *unit circle*
  - In $\mathbf{R}^3$, all unit vectors lie on the surface of the *unit sphere*
  - In $\mathbf{R}^n$, we will say that all unit vectors like on the surface of the *unit ball*





Normalization of vectors

- Given a non-zero vector **u**,

  we can define its associated unit vector by defining

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

  – Again, in general, we will almost exclusively use the 2-norm

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|_2}$$

Normalization of vectors

- For example, given $\mathbf{u} = \begin{pmatrix} 1 \\ -10 \\ 5 \\ 9 \\ 7 \end{pmatrix}$

  we note $\|\mathbf{u}\|_1 = 1 + 10 + 5 + 9 + 7 = 32$,
  so normalizing $\mathbf{u}$ with respect to the 1-norm is $\hat{\mathbf{u}} = \begin{pmatrix} 0.03125 \\ -0.3125 \\ 0.15625 \\ 0.28125 \\ 0.21875 \end{pmatrix}$

Normalization of vectors

- For example, given $\mathbf{u} = \begin{pmatrix} 1 \\ -10 \\ 5 \\ 9 \\ 7 \end{pmatrix}$

we note $\|\mathbf{u}\|_2 = \sqrt{1+100+25+81+49}$

$$= \sqrt{1+100+25+81+49}$$

$$= \sqrt{256} = 16$$

so normalizing **u** with respect to the 2-norm is $\hat{\mathbf{u}} = \begin{pmatrix} 0.0625 \\ -0.625 \\ 0.3125 \\ 0.5625 \\ 0.4375 \end{pmatrix}$

Normalization of vectors

- For example, given $\mathbf{u} = \begin{pmatrix} 1 \\ -10 \\ 5 \\ 9 \\ 7 \end{pmatrix}$

  we note $\|\mathbf{u}\|_\infty = \max\{1, 10, 5, 9, 7\} = 10$,
  so normalizing $\mathbf{u}$ with respect to the ∞-norm is $\hat{\mathbf{u}} = \begin{pmatrix} 0.1 \\ -1 \\ 0.5 \\ 0.9 \\ 0.7 \end{pmatrix}$

In this course…

- In general, in this course, we will be using the 2-norm and any normalizing will be performed with respect to this 2-norm
- For example, if $\mathbf{u} = \begin{pmatrix} -0.6 \\ 1.8 \\ -1.2 \\ -1.1 \end{pmatrix}$, $\|\mathbf{u}\|_2 = \sqrt{0.36 + 3.24 + 1.44 + 1.21} = \sqrt{6.25} = 2.5$
- Thus, $\hat{\mathbf{u}} = \begin{pmatrix} -0.24 \\ 0.72 \\ -0.48 \\ -0.44 \end{pmatrix}$

Summary

- Following this topic, you now
  - Understand the term *unit vector*
  - Know that we will be generally using the 2-norm
  - Understand the algorithm for converting a non-zero vector $\mathbf{u}$ into its associated unit vector $\hat{\mathbf{u}}$

References

[1] https://en.wikipedia.org/wiki/Norm_(mathematics)

[2] https://en.wikipedia.org/wiki/Unit_vector

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.

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